Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **17**, No. 6, pp. 1591-1624

DOI: 10.4208/aamm.OA-2024-0037
December 2025

Improved Sixth-Order WENO Finite Difference Schemes for Hyperbolic Conservation Laws*

Cai-Feng Wang[1], Wai-Sun Don[1,2], Jia-Le Li[1] and Bao-Shan Wang[1,*]

[1] *School of Mathematical Sciences, Ocean University of China, Qingdao, Shandong 266100, China*
[2] *Department of Mathematics, Hong Kong Baptist University, Hong Kong, China*

Received 10 February 2024; Accepted (in revised version) 18 July 2024

Abstract. This article describes developing and improving sixth-order characteristic-wise Weighted Essentially Non-Oscillatory (WENO) finite difference schemes. These schemes are specially designed to solve scalar and system hyperbolic conservation laws with high accuracy/resolution and robustness. The schemes have been enhanced by using a new reference global smoothness indicator, which ensures the optimal order of accuracy for smooth solutions. The schemes also incorporate affine-invariant nonlinear Ai-weights that are independent of the scaling of solution and the choice of sensitivity parameter. The improved nonlinear weights enhance the essentially non-oscillatory (ENO) capturing of discontinuities and minimize the numerical dissipation, especially for long-time simulations. The study also introduces the positivity-preserving limiter to ensure that the numerical solution of Euler equations is physically valid. The effectiveness of improved schemes is demonstrated through one- and two-dimensional benchmark shock-tube problems, such as the Sod, Lax, and Woodward-Colella problems. The improved schemes are compared with other WENO schemes in terms of accuracy, resolution, ENO, and robustness.

AMS subject classifications: 35L65, 65M06, 76M20

Key words: Ai-WENO, critical points, global smoothness indicator, low dissipation, positivity-preserving, long-time simulation.

1 Introduction

Hyperbolic conservation laws are commonly solved numerically using the characteristic-wise weighted essential non-oscillatory (WENO) finite difference schemes. These schemes

*All the authors have contributed equally to this scholarly research and article.
*Corresponding author.
Emails: wangcaifeng@stu.ouc.edu.cn (C.-F. Wang), donwaisun@outlook.com (W.-S. Don), lijiale@stu.ouc.edu.cn (J.-L. Li), wbs@ouc.edu.cn (B.-S. Wang)

are favored for their ability to capture shocks in a non-oscillatory manner and efficiently resolve smooth, small, weak structures. The first WENO scheme was proposed by Liu et al. [23], which achieved $(r+1)$ order of accuracy using a convex combination of rth ENO reconstructions. Jiang and Shu [17] later improved the WENO scheme by proposing a general framework for designing smoothness indicators and nonlinear weights. Despite its advantages, the fifth-order WENO-JS5 scheme only achieved third-order accuracy at first-order critical points. To address this, Henrick et al. [14] derived the necessary conditions for the nonlinear weights to achieve optimal order convergence and designed a mapping function for the nonlinear weights. This resulted in the WENO-M scheme, which achieved a fifth-order convergence rate at critical points. Borges et al. [5] further improved this by designing an optimal order global smoothness measurement to build new nonlinear Z-type weights, resulting in the WENO-Z scheme. The WENO-Z scheme has been widely recognized for its accuracy, resolution, shock-capturing, and overall efficiency [5,16].

The WENO schemes mentioned above use an upwind-biased global stencil. To reduce the excessive dissipation of the classical WENO-JS5 scheme, a central global stencil is adopted to develop low-dissipation WENO schemes. Martin et al. [26] developed global stencil-symmetric WENO-SYMOO and WENO-SYMBO schemes by adding an extra downwind candidate stencil. Hu et al. [12] proposed the adaptive central-upwind sixth-order WENO-CU6 scheme, which adapts smoothly between the sixth-order central scheme in smooth regions and the fifth-order upwind WENO scheme near discontinuities. The WENO-CU6 scheme generates numerical oscillations near discontinuities with increasing grid numbers and a larger CFL number. To mitigate this, Hu et al. [12] replaced the three-point downwind substencil with a four-point one to include upwind information and increase stability. They developed the WENO-M6 and WENO-Z6 schemes using the ideas of mapping in the WENO-M [14] and the Z-type weights in the WENO-Z [4] schemes, respectively. The WENO-Z6 scheme achieves fifth-order accuracy at critical points and is more robust and efficient overall than the WENO-CU6 scheme.

In the WENO polynomial reconstruction (WPR) procedure, a convex combination of several low-order polynomial reconstructions is used to obtain an optimal high-order reconstruction of a function. The linear weights for this combination are determined based on a relationship between the local low-order and global high-order reconstructions. Huang and Chen [15] introduced a new adaptive central-upwind sixth-order WENO6-S (or WENO-S6) scheme that uses a convex combination of a global sixth-order and three local third-order reconstructions. The linear weights are chosen arbitrarily under certain constraints. This scheme can accurately reconstruct smooth functions but not those with high-order critical points. Numerical tests have shown it to be more efficient than other schemes, such as WENO-JS5 and WENO-CU6, with the same CFL number. Zhao et al. [37] developed the WENO-CZ6 scheme, which uses a convex combination of a global sixth-order and four local low-order reconstructions. Numerical tests have shown that this scheme has a higher resolution than the WENO-Z6 scheme and is more efficient than the WENO-S6 scheme. However, classical WENO operators such as WENO-S6/CZ6 can-