Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **17**, No. 6, pp. 1682-1714

DOI: 10.4208/aamm.OA-2024-0026
December 2025

An Explicit Method for the Coupled Forward Backward Stochastic Differential Equations

Xiao Tang[1,*] and Jie Xiong[2]

[1] *School of Mathematics and Computational Science and Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Xiangtan University, Xiangtan, Hunan 411105, China*
[2] *Department of Mathematics and Shenzhen International Center for Mathematics, Southern University of Science and Technology, Shenzhen, Guangdong 518055, China*

Received 6 February 2024; Accepted (in revised version) 29 September 2024

Abstract. This paper proposes an explicit method for the coupled forward backward stochastic differential equations (FBSDEs). Our method combines skillfully a weak second order stochastic Runge-Kutta method for solving forward equations with a two-step method for solving backward equations. We give a convergence theorem for the proposed method when the FBSDEs is weakly coupled (the forward equations are independent of the variable Z). Finally, some numerical results are presented, and the numerical results show that our method still works well even if the forward equations depend on Z.

AMS subject classifications: 65C30, 60H10, 65L06

Key words: Forward backward stochastic differential equations, explicit method, convergence.

1 Introduction

Let T denote a given positive number and $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{0\le t\le T})$ be a filtered probability space, where $\{\mathcal{F}_t\}_{0\le t\le T}$ is the natural filtration generated by a standard m-dimensional Wiener process $W(t)=(W_1(t),W_2(t),\cdots,W_m(t))^\top$ (The superscript $\top$ denotes the transpose). Consider the coupled forward backward stochastic differential equation (FBSDE) of the integral form

$$X(t)=X(0)+\int_0^t b\big(s,X(s),Y(s),Z(s)\big)ds+\int_0^t \sigma\big(s,X(s),Y(s)\big)dW(s), \tag{1.1a}$$

$$Y(t)=\varphi\big(X(T)\big)+\int_t^T f\big(s,X(s),Y(s),Z(s)\big)ds-\int_t^T Z(s)dW(s), \quad t\in[0,T], \tag{1.1b}$$

*Corresponding author.
Emails: tangx@xtu.edu.cn, 1293060753@qq.com (X. Tang), xiongj@sustech.edu.cn (J. Xiong)

where $X(0)\in\mathbb{R}^{d\times 1}$, the functions $b:[0,T]\times\mathbb{R}^{d\times 1}\times\mathbb{R}\times\mathbb{R}^{1\times m}\longrightarrow\mathbb{R}^{d\times 1}$, $\sigma:[0,T]\times\mathbb{R}^{d\times 1}\times\mathbb{R}\longrightarrow\mathbb{R}^{d\times m}$, $f:[0,T]\times\mathbb{R}^{d\times 1}\times\mathbb{R}\times\mathbb{R}^{1\times m}\longrightarrow\mathbb{R}$ and $\varphi:\mathbb{R}^{d\times 1}\longrightarrow\mathbb{R}$. Since a study of the regularity of the solution is not at all the aim of this paper, for convenience, we assume that the functions b, σ, f, φ satisfy the following Assumption 1.1 so that the solution of (1.1) has good enough regularity.

Assumption 1.1. The functions b, σ, f, φ are smooth enough and all these functions together with their partial derivatives up to an appropriate order are uniformly bounded. In addition, the function $\sigma\geq\sigma_0$, where σ_0 is a positive constant.

Under Assumption 1.1, the solution of the FBSDE (1.1) is connected with a terminal value Cauchy problem (see, e.g., [10–17]). More precisely, the solutions $Y(t)$, $Z(t)$ of (1.1) can be rewritten as

$$Y(t)=u\big(t,X(t)\big),\quad Z(t)=\nabla u\big(t,X(t)\big)\sigma\big(t,X(t),u\big(t,X(t)\big)\big), \tag{1.2}$$

where ∇u denotes the gradient of $u(t,x)$ with respect to x, and $u(t,x)$ is the classical solution of the following quasi-linear parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t}+\displaystyle\sum_{i=1}^{d}b_i\big(t,x,u,\nabla u\sigma(t,x,u)\big)\dfrac{\partial u}{\partial x_i}+\dfrac{1}{2}\sum_{i,j=1}^{d}g_{ij}(t,x,u)\dfrac{\partial^2 u}{\partial x_i\partial x_j} \\ =-f\big(t,x,u,\nabla u\sigma(t,x,u)\big),\quad t\in[0,T), \\ u(T,x)=\varphi(x),\quad x\in\mathbb{R}^d, \end{cases} \tag{1.3}$$

with $g_{ij}=\sum_{k=1}^{m}\sigma_{ik}\sigma_{jk}$.

The FBSDEs play an important role in stochastic control theory and mathematical finance (see, e.g., [5,11,25]). Unfortunately, it is very difficult to find the analytic solution to most FBSDEs. Therefore, developing efficient numerical methods for solving the FBSDEs is becoming highly desired in practical applications.

If $X(t)=W(t)$, the equation (1.1) degenerates to the backward stochastic differential equation (BSDE). This is the simplest case. Up to now, many works on the numerical methods for the BSDEs and the related Cauchy problem have been done (see, e.g., [26, 27, 31] and references therein). If the functions b,σ do not depend on $Y(t)$ and $Z(t)$, the equation (1.1) is called to be decoupled FBSEDs. In this case, there are also many good numerical methods (see, e.g., [2,3,23,24,29,30] and references therein).

For the coupled FBSDE (1.1), the related results on numerical methods are much less than those of the above two special cases. In the general case of equation (1.1), the Euler method is the most frequently used method (see, e.g., [1, 4, 12, 13]), which is relatively simple for analysis and implementation. However, it has a lower rate of convergence. It is worth mentioning that the authors of references [7, 9, 20, 21, 28] have proposed some methods, which can realize high order for the coupled FBSDEs or mean-field FBSDEs. Numerical results show that those methods in [7, 9, 21, 28] are very efficient. It is a pity