Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **17**, No. 6, pp. 1715-1741

DOI: 10.4208/aamm.OA-2024-0128
December 2025

Mechanisms for Stabilizing Thermal Lattice Boltzmann Equation and Their Applications for Convection-Diffusion Problem on Nonuniform Meshes

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Received 12 June 2024; Accepted (in revised version) 23 February 2025

Abstract. The thermal lattice Boltzmann equation (TLBE) has superior numerical stability as an explicit algorithm. However, its applications on nonuniform meshes are complicated. This paper clarifies the intrinsic mechanism for stabilizing computations in TLBE and proposes two solvers that combine the numerical stability of TLBE and flexible finite difference/volume schemes for nonuniform meshes. Through a brief review of the lattice Boltzmann method, it is concluded that the entropy increase of the collision operator is essential for numerical stability. This paper first proposes a macroscopic entropy-increasing (MEI) model for convection-diffusion problems by combining the MEI process and TLBE. The von Neumann stability analysis proves that the MEI model has no upper limit for mesh Fourier number. However, the accuracy of the MEI model is found to be sensitive to higher-order deviation terms. Therefore, a hybrid model that combines the MEI and the equilibrium-moment-based models is proposed to solve the problem. The von Neumann stability analysis demonstrates that the hybrid model can completely recover the numerical stability of TLBE. Numerical investigations validate the good stability and accuracy of the hybrid model. Most importantly, it can be easily applied to nonuniform meshes, whereas implementing TLBE on nonuniform meshes is relatively complicated.

AMS subject classifications: 65M22, 35B35, 76R50, 35Q20

Key words: Thermal lattice Boltzmann equation, numerical stability, entropy, nonuniform mesh, finite volume scheme.

1 Introduction

The Boltzmann equation provides a general mesoscopic description of non-equilibrium physical processes, such as fluid flow and heat/mass transfer. Based on approximations

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of the Boltzmann equation, the lattice Boltzmann method (LBM) [1] was proposed to simulate incompressible flow. Due to its simple implementation and good numerical stability, LBM has received much attention. Subsequently, a series of lattice Boltzmann models have been proposed to simulate heat/mass transfer [2–4], multiphase flow [5–7], shallow wave [8], and so on.

The solution of the lattice Boltzmann equation (LBE) consists of two steps: the local collision and non-local streaming. To achieve simple implementation, the mesh is designed to be uniform in general to ensure that the streaming process always proceeds on computational nodes. The spatial discretization scheme highly relates to its numerical stability and low dissipation, however, it simultaneously brings complexity to usage on nonuniform meshes. Many attempts have been made to extend LBM's applications to nonuniform meshes.

The first approach is interpolation [9, 10]. In this approach, LBM is implemented like that on a uniform mesh, and required distribution functions at specified positions are obtained by interpolations. This approach applies to general nonuniform meshes but may have a heavy burden for 3D simulations.

The second approach is constructing LBE for the structured nonuniform mesh [11–14]. This kind of model has been proven to perform well on structured meshes. The extension to general unstructured meshes has not been reported due to the complexity.

The third approach is to solve the discrete Boltzmann equation with finite volume/element methods [15–18]. Compared with the standard LBE, it has an obvious advantage in tackling complex geometry. On the other hand, n equations need to be solved separately for a DmQn (m dimensions and n discrete velocities) model, which increases the complexity and computational time cost.

The fourth approach is the lattice Boltzmann flux solver [19, 20]. It applies the finite volume scheme to update macroscopic variables at cell centers and constructs a simplified LBM on the cell face to evaluate interface fluxes. After detailed analyses [21, 22], it was proven that the lattice Boltzmann flux solver has enhanced numerical stability at small relaxation times but cannot ensure stability at large relaxation times.

The fifth approach is reconstructing LBE as moment equations and solving them with finite difference/finite volume schemes. The key point is that the mechanism of the good numerical stability of LBM must be well understood and recovered in the reconstructed equations. Many reconstructions [23, 24] focus on recovering continuous moment equations. Because LBM is a discrete algorithm, the mechanisms for stabilizing computations are lost in continuous moment equations. Some reconstructions [25–27] pointed out that the dissipation terms in time-discretized conservative moment equations are essential to stabilize computations. More detailed numerical investigations [28] demonstrate that these dissipation terms are insufficient to interpret the numerical stability at large relaxation times. These facts imply that LBM has an intrinsic mechanism for stabilizing computations at large relaxation times, and it has not been clarified yet. The present paper aims to clarify the intrinsic mechanism and then combine LBM's numerical stability/accuracy and flexible finite difference/volume scheme for nonuniform meshes. For