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# On Convergence and Superconvergence of Discontinuous Galerkin Method for Semi-Explicit Index-1 Integro-Differential Algebraic Equations

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**Abstract.** This paper mainly focuses on the discontinuous Galerkin (DG) method for solving the semi-explicit index-1 integro-differential algebraic equation (IDAE), which is a coupled system of Volterra integro-differential equations (VIDEs) and second-kind Volterra integral equations (VIEs). The DG approach is applied to both the VIDE and VIE components of the system. The global convergence respectively in the $L^2$-norm and $L^\infty$-norm is established, and the local superconvergence for VIDE component is obtained. Furthermore, numerical examples are presented to validate the theoretical convergence and superconvergence results.

**AMS subject classifications**: 45J05, 65R20

**Key words**: Integro-differential algebraic equations, index 1, DG method, convergence, superconvergence.

## 1 Introduction

In our earlier work [29], we consider the following semi-explicit index-1 integro-differential algebraic equation (IDAE):

$$\begin{cases} x_1'(t)+b_{11}(t)x_1(t)+b_{12}(t)x_2(t)=f_1(t)+\displaystyle\int_0^t [K_{11}(t,s)x_1(s)+K_{12}(t,s)x_2(s)]ds, \\ b_{21}(t)x_1(t)+b_{22}(t)x_2(t)=f_2(t)+\displaystyle\int_0^t [K_{21}(t,s)x_1(s)+K_{22}(t,s)x_2(s)]ds, \end{cases} \tag{1.1}$$

with $t\in I:=[0,T]$. Here, the given functions $f_p$, $b_{pq}$, $K_{pq}$ $(p,q=1,2)$ are smooth in $I$ and $D:=\{(t,s):0\le s\le t\le T\}$, respectively, and $|b_{22}(t)|\ge b_0>0$. The system (1.1) requires a consistent condition

$$b_{21}(0)x_1(0)+b_{22}(0)x_2(0)=f_2(0)$$

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imposed by the initial values $(x_1(0), x_2(0))^T = (x_1^0, x_2^0)^T$. In [29], a mixed Galerkin method is studied for the semi-explicit index-1 IDAE (1.1), where the VIDE component of the IDAE is approximated by the continuous Galerkin (CG) method, and the VIE component is approximated by the discontinuous Galerkin (DG) method.

However, the IDAE model encountered in practical applications is generally not appeared in a semi-explicit form, which means that we can not recognize the VIDE and VIE components in advance. In this case, it is not possible to apply two different numerical methods for two different components. On the other hand, the VIDEs can be viewed as ordinary differential equations (ODEs) with a memory term. The results in [19, 25, 26] indicate that the memory term does not affect the convergence order. In addition, the DG method has several advantages over the CG method for ODEs as in [1,5,8,11,12,14,22]: (i) It achieves a higher order of convergence (from $2k$ to $2k+1$ by using the same polynomial approximate spaces with degree $k$); (ii) The solution is allowed to be discontinuous across the entire interval, providing greater flexibility. Thus, this is the motivation to investigate the IDAE (1.1) by using DG method for both VIDE and VIE components. What is more, the $L^\infty$ error estimation is given in [29], but the $L^2$ error estimation is not considered. In this paper, we give the DG error estimations for both $L^\infty$ and $L^2$ norms.

IDAEs have many applications in physics and engineering, including the modeling of Kirchhoff's laws [9, 10], seat-occupant dynamics [13], and hydraulic circuits in combustion processes [20]. Due to the existence of the memory term, the exact solution of IDAEs is difficult to obtain; hence, it is important to investigate effective numerical methods. The numerical method for IDAEs was first studied by Kauthen [15], who analyzed the convergence of implicit Runge-Kutta methods. Since then, many numerical methods have been proposed for solving IDAEs, including collocation methods [3, 18, 28], multistep methods [6, 7], spectral methods [21], and $\Sigma$ method [31].

**Notations**: Let $a,b\in\mathbb{R}$ with $a<b$. The standard $L^2$-norm of an integrable function $u$ in $[a,b]$ is defined as $\|u\|_{[a,b],0} := (\int_a^b u^2(t)dt)^{\frac{1}{2}}$. Furthermore, the standard $L^\infty$-norm of $u$ in $[a,b]$ is given by $\|u\|_{[a,b],\infty} := \operatorname*{esssup}_{t\in[a,b]} |u(t)|$. Let $p\geq 1$ be an integer. The standard Sobolev spaces $H^p[a,b]$ consists of functions having generalized derivatives of order $p$ in the space $L^2[a,b]$, i.e.,

$$H^p[a,b] := \left\{ u\in L^2[a,b] : u^{(k)} = \frac{d^k u}{dt^k} \in L^2[a,b],\ \forall k\leq p \right\}.$$

The norm of $H^p[a,b]$ is defined as

$$\|u\|_{[a,b],p} := \left( \sum_{k=0}^{p} \|u^{(k)}(t)\|^2_{[a,b],0} \right)^{\frac{1}{2}}$$

and the semi-norm as

$$|u|_{[a,b],p} := \|u^{(p)}(t)\|_{[a,b],0} \quad \text{and} \quad |u|_{[a,b],p,\infty} := \|u^{(p)}(t)\|_{[a,b],\infty}.$$