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A Primal-Dual Discontinuous Galerkin Finite Element Method for Ill-Posed Elliptic Cauchy Problems

Yanli Chen[1], Tie Zhang[2,*] and Ying Sheng[1]

[1] *Department of Mathematics, Northeastern University, Shenyang, Liaoning 110004, China*
[2] *Department of Mathematics and the State Key Laboratory of Synthetical Automation for Process Industries, Northeastern University, Shenyang, Liaoning 110004, China*

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Abstract. We present a primal-dual discontinuous Galerkin finite element method for a type of ill-posed elliptic Cauchy problem. It is shown that the discrete problem attains a unique solution, if the solution of the ill-posed elliptic Cauchy problems is unique. An optimal error estimate is obtained in a H^1-like norm. Numerical experiments are provided to demonstrate the efficiency of the proposed method.

AMS subject classifications: 65N15, 65N30

Key words: The ill-posed elliptic problem, discontinuous Galerkin method, primal-dual scheme, optimal error estimate.

1 Introduction

In this paper, we consider the following ill-posed elliptic Cauchy problem

$$\begin{cases} -\nabla\cdot(a\nabla u)=f & \text{in } \Omega, \\ u=g_1 & \text{on } \Gamma_d, \\ (a\nabla u)\cdot\boldsymbol{n}=g_2 & \text{on } \Gamma_n, \end{cases} \tag{1.1}$$

where Ω is a bounded polygonal or polyhedral domain in $\mathbb{R}^d(d=2,3)$ with Lipschitz continuous boundary $\partial\Omega$, Γ_d and Γ_n are polygonal subsets of the boundary $\partial\Omega$, $\boldsymbol{n}$ is a unit outward normal direction to $\partial\Omega$, $f\in L^2(\Omega)$. The coefficient $a(\boldsymbol{x})\in W^{1,\infty}(\Omega)$ is assumed to be bounded in Ω, i.e., there exist positive constants $a_{\min}$ and $a_{\max}$ such that $a_{\min}\le a(\boldsymbol{x})\le a_{\max}, x\in\Omega$. The boundary data g_1 and g_2 are two given functions defined on Γ_d and Γ_n,

*Corresponding author.
Emails: chenyanli@mail.neu.edu.cn (Y. Chen), ztmath@163.com (T. Zhang), shengying@mail.neu.edu.cn (Y. Sheng)

respectively. We denote the complement of the Neumann boundary by $\Gamma_n^c := \partial\Omega \setminus \Gamma_n$. We assume that problem (1.1) is ill-posed, that is, $\Gamma_d \cap \Gamma_n \neq \emptyset$ or $\Gamma_d \cup \Gamma_n \neq \partial\Omega$.

Contrary to a well-posed elliptic problem, the ill-posedness of the problem (1.1) results from some special practical applications, where the Dirichlet data g_1 and the Neumann data g_2 are both available on a common part of domain boundary (i.e., $\Gamma_d \cap \Gamma_n \neq \emptyset$), and the boundary conditions or its data may be lost on a part of domain boundary (i.e., $\Gamma_d \cup \Gamma_n \neq \partial\Omega$). For instance, the elliptic Cauchy problem plays a crucial role to use the electrical impedance tomography for noninvasive detection [7, 23]. In this application, a weak current is applied to the electrodes on the surface of the human body and then the voltage values on the electrodes is measured. It means that $\Gamma_d = \Gamma_n$ in the model problem. According to the relationship between the voltage on Γ_d and the current on Γ_n, the internal electrical impedance of the human body or the change value of electrical impedance can be reconstructed. For more applications and relative results on elliptic Cauchy problem we refer to [2, 9–11, 19, 20, 22] and the references cited therein.

It is well known that the Cauchy problem defined as (1.1) is severely ill-posed [6] and even when a solution exists, it does not depend continuously on the boundary data. Therefore, how to design accuracy computational methods for approximating the ill-posed elliptic Cauchy problem remains a challenging topic. Most numerical methods are designed based on the well-posedness of the physical model problem. Following this basic idea, an important strategy for numerically approximating the elliptic Cauchy problem is to regularize the ill-posed problem to obtain a well-posed problem. The reduced well-posed problem then can be solved numerically using standard approximation techniques, such as finite element methods (FEM), boundary element methods and hybrid methods. We refer to [3, 4, 8, 12, 13, 24] for regularization methods and related approximation on the ill-posed Cauchy problem.

In 2013, Burman introduced the primal-dual stabilized finite element methods for ill-posed elliptic problem in [5]. This primal-dual method discretized the ill-posed problem through a constrained optimization problem. The unstable discrete problem was then stabilized by using techniques known from the theory of the stabilized finite element methods. Later, the method was further developed for the approximation of elliptic data assimilation problems [14], parabolic data reconstruction problems [15, 16], well-posed convection-diffusion problems [18]. As an extension of this method, a primal-dual weak Galerkin finite element methods were proposed in [25, 26] by employing weak finite element functions to approximate the solution of the elliptic Cauchy problem.

Motivated by the work of Burman [5], we develop a new primal-dual method which is based on the discontinuous Galerkin finite element spaces. Although the idea in [5] can be applied to any finite dimensional space, we point out that our techniques completely differ from Burman [5] in the numerical analysis. Specifically, the convergence analysis in [5] is based on the conditional stability estimates for the exact solution of Cauchy problem, while our analysis is derived by the consistency of the discrete scheme (in the sense that the exact solution satisfies the discrete system) and a special norm on the discontinuous Galerkin finite element spaces. One of our main results indicates that