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Modified Galerkin Method for Derivative Dependent Fredholm–Hammerstein Integral Equations of Second Kind

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Abstract. In this paper, we consider modified Galerkin and iterated modified Galerkin methods for solving a class of two point boundary value problems. The methods are applied after constructing the equivalent derivative dependent Fredholm-Hammerstein integral equations to the boundary value problem. Existence and convergence of the approximate solutions to the actual solution is discussed and the rates of convergence are obtained. Superconvergence results for the approximate and iterated approximate solutions of piecewise polynomial based modified Galerkin method in infinity norm are given. We have also established that iterated modified Galerkin approximation improves over the modified Galerkin solution. Numerical examples are presented to illustrate the theoretical results.

AMS subject classifications: 45B05, 45G10, 65R20

Key words: Fredholm integral equations, Green's kernel, modified Galerkin method, piecewise polynomial, superconvergence rates.

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1 Introduction

Consider the following two-point boundary value problem

$$(\vartheta'(t))' = \phi(t,\vartheta(t),\vartheta'(t)), \tag{1.1}$$

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subject to the boundary conditions

$$\vartheta(0)=\alpha_1, \quad \beta_1\vartheta(1)+\gamma_1\vartheta'(1)=\eta_1.$$

Frequently, different phenomena in scientific fields, including mechanics, optimization, communication theory, fluid mechanics, electricity, magnetism, and many other applied science problems are reduced to solve the boundary value problem (1.1). The numerical treatment of the above boundary value problems has always been far from trivial. The following integral equations arise as reformulation of the above type singular two-point boundary value problem (1.1)

$$\vartheta(t)=\alpha_1+\frac{(\eta_1-\alpha_1\beta_1)t}{\beta_1+\gamma_1}+\int_0^1\kappa(t,\chi)\phi(\chi,\vartheta(\chi),\vartheta'(\chi))d\chi, \quad 0\le t\le 1, \tag{1.2}$$

where $\kappa(t,\chi)$ is given by

$$\kappa(t,\chi)=\begin{cases} t\Big(1-\dfrac{\beta_1\chi}{\beta_1+\gamma_1}\Big), & 0\le t\le\chi, \\ \chi\Big(1-\dfrac{\beta_1 t}{\beta_1+\gamma_1}\Big), & \chi\le t\le 1. \end{cases}$$

The main difficulty of (1.1) is that the singularity behavior occurs at $t=0$. With the use of important properties of Green's functions, it would be easier to handle these equations after constructing the equivalent nonlinear Fredholm integral equations. The same is mentioned in [27], where authors discussed numerical solvability of the similar kind of singular two-point boundary value problem after reformulating them into a nonlinear Fredholm integral equation with Green's kernel. Also, with this reformultaion, the higher order derivative approximation for (1.1) can be avoided, which is computationally very much favorable. In the last few years, effective methods such as decomposition method, the Adomian decomposition method, and the modified decomposition method etc. are developed for numerically solving different types of boundary value problems and associated integral equations (see [1,6,7,15,16,27]). In attempt of improving the accuracy of the approximate solutions, projection methods are used to solve Fredholm integral equations. Several results on different projection methods to solve nonlinear Fredholm integral equations can be found in literature (see [11,12,17,19,21,22,25,26]). Classical projection methods such as Galerkin, collocation methods for Fredholm Hammerstein integral equations with smooth as well as weakly singular kernels were developed and superconvergence was obtained by several authors (see [9,11,12,17–19,21,22,26]) etc.). Piecewise polynomial based Galerkin method is applied to investigate the approximate solutions of nonlinear Fredholm-Hammerstein integral equations with smooth kernels in [9]. Authors developed projection and iterated projection methods to solve nonlinear Fredholm-integral equation with particular classes of kernels having singularity (see [3–5]).

In literature, many attempts have been made to improve the accuracy of numerical solutions of different integral equations using projection methods. In [20], authors created the modified projection method and showed that under the same assumptions of