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On Field Concentration between Nearly-Touching Multiscale Inclusions in the Quasi-Static Regime

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Abstract. Strong field concentration may occur between two nearly-touching high-contrast material inclusions due to an incident field. The degree of concentration is characterised by the blowup rate of the underlying gradient field. This phenomenon has received considerable attention in the literature due to its practical implications in the theory of composite materials. However, most of the existing studies are concerned with the static cases. In this paper, we present a comprehensive numerical investigation of this intriguing phenomenon associated to the Helmholtz system in the quasi-static frequency regime. On the one hand, we present extensive numerical results to corroborate the theoretical findings in [16], and on the other hand, we derive new findings that cannot be handled by the theoretical analysis yet. Our focus is on the static effect, frequency effect, the geometric (curvature) effect to the gradient estimates.

AMS subject classifications: 35J25, 35C20, 78A40

Key words: Field concentration, gradient estimates, Helmholtz system, high-contrast inclusions, blow up, quasi-static, multi-scale, curvature.

1 Introduction

Strong field concentration may occur between two nearly-touching high-contrast material inclusions due to an incident field. The degree of concentration is characterised by the blowup rate of the underlying gradient field [7]. This phenomenon has received considerable attention in the literature due its practical implications in the theory of composite materials, where the building blocks are generally high-contrast material inclusions; see [12, 31, 32] for related studies for general elliptic systems; [2, 9, 10, 19, 25, 28, 41] for the Lamé system; [3] for the stokes flow problem; [4, 5, 8, 27, 29, 33, 40] for the electro-static field, and recently [16, 17] for the wave field in the quasi-static regime.

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In the electrostatic case, the gradient blowup rate for perfect conductors was shown to be $1/\sqrt{\epsilon}$ in two dimensions in [5,40] and $(\epsilon|\ln\epsilon|)^{-1}$ in three dimensions in [8]. Here, $\epsilon\in\mathbb{R}_+$ signifies the asymptotic distance parameter between the close-to-touching inclusions. In the linear elasticity Lamé system, the blowup rate of the stress in a narrow domain between two stiff inclusions with infinite shear modulus is $1/\sqrt{\epsilon}$ in two dimensions in [9,28]. In [10], it is proved that the upper bound of the blow up rate of the strain tensor is $(\epsilon|\ln\epsilon|)^{-1}$ in dimension three, and is ϵ^{-1} in the dimension $d>4$. In the two-dimensional Stokes flow, it is proved in [3] that either the pressure or the shear stress component of the stress tensor blows up, thus the stress always blows up as the distance between the two cylinders tends to zero, and the blow-up rate is $1/\sqrt{\epsilon}$.

As discussed above, most of the existing studies in the literature focus on the inclusions that are of regular size, i.e., the size is of order $\mathcal{O}(1)$ compared to the asymptotic distance parameter $\epsilon\ll 1$, and moreover most studies focus on the static scenarios, namely the frequency is formally taken to be zero. However, in the state-of-the-art development on composite materials, one needs to consider the quasi-static wave fields interacting with closely spaced high-contrast material inclusions, especially for the effective construction of the so-called metamaterials; see e.g., [6,21,30,37,38] and the references cited therein. Hence, it is natural to consider the field concentration in such a new scenario which may produce important practical implications to the composite construction of metamterials, say e.g., the stability of the effective metamaterial structures. Recently, the authors in [16,17] considered the field concentration between closely spaced high-contrast inclusions associated with the Helmholtz system in the quasi-static regime, and derived accurate quantitative characterisations of the gradient fields. However, only radial inclusions were analyzed in [16,17], though they may be of different scales in terms of the asymptotic distance parameter ϵ.

In this paper, we numerically investigate the gradient field estimates for the Helmhotlz system associated with multiscale high-contrast nearly-touching material inclusions in the quasi-static regime. We focus on studying the frequency effect, the multiscale effect and the geometric (especially, the curvature) effect to the field concentration. Roughly summarising, it is found that the frequency effect can induce new field concentration phenomena (not due to the sub-wavelength resonance), and the size scales of the inclusions can either enhance or reduce the field concentration. In fact, the size scale is related to the curvature effect on the field concentration. Based on numerical observations, we show new discoveries on the curvature effect in more general geometric setups. On the one hand, our numerics corroborate the theoretical results in [16,17], which in turn benchmarks the effectiveness and efficiency of our numerical method, and on the other hand, they enable us to derive new findings that cannot be handled by the theoretical analysis yet.

The rest of the paper is organized as follows. In the next section we present the mathematical setup of our study. In Section 3, we conduct extensive numerical experiments to quantitatively verify the characterizations of the gradient fields in the radial cases. Section 4 is devoted to new findings in the general geometric setups. The paper is concluded with some relevant discussions in Section 5.