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A Novel Temporal Two-Grid Compact Finite Difference Scheme for the Viscous Burgers' Equation

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Abstract. We present a novel two-grid compact finite difference scheme for the viscous Burgers' equation in this paper, where the second-order Crank-Nicolson method is used to deal with the time marching, the compact finite difference formula is used to approximate the spatial second-order term, and the nonlinear convection term is discretized using the developed nonlinear fourth-order operator, providing the scheme with both high fourth-order spatial convergence and a low computational cost. The scheme is then established in three steps, with the first step being the construction of a nonlinear coarse-grid compact finite difference scheme that is solved iteratively using a fixed point iterative method, the second step being the application of the Lagrange interpolation formula to obtain a rough solution on the fine grid, and the third step being the development of the linearized fine-grid compact finite difference scheme. We also perform a convergence and stability analysis on the developed scheme, and the results show that the scheme can achieve spatial fourth-order and temporal second-order convergence. Finally, a number of numerical examples are provided to validate the theoretical predictions.

AMS subject classifications: 65M06, 65M22, 65N12

Key words: Two-grid, compact finite difference, viscous Burgers, stability, error analysis.

1 Introduction

Burgers' equation, also known as the Bateman-Burgers equation [2], is a fundamental partial differential equation (PDE) that can be found in many areas of applied mathematics, including shock waves [21], gas dynamics [4, 19], and fluid mechanics [3, 41], etc. On the level of theoretical research, the viscous Burgers' equation is a semilinear

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parabolic equation similar to the Navier-Stokes (N-S) equations. It differs from the N-S equations only in a pressure term. The N-S equation is of great significance in fluid mechanics. About the recent development of numerical analysis for the N-S equations, we refer to [6, 7, 22–26, 40]. As for Burgers' equation, its analytical solution can be obtained using the Hopf-Cole transformation [13, 17], but its complex form prevents it from being used in engineering applications. As a result, developing a numerical method with high precision and stability to numerically approximate the Burgers' equation has become an unavoidable alternative to replace the complicated analytical solution. It is remarkable that the numerical schemes for solving the Burgers' equation have been developed extensively, ranging from the spectral method [15, 16], finite element method [12, 29, 34], finite difference method [32, 36], cubic B-splines method [10], to the spatial two-grid method [18], and so on. In addition, there are also a few studies that focus on various boundary conditions that are equipped with Burgers' equation, such as the Neumann boundary [20] or Robin boundary conditions [37].

The purpose of this paper is to develop a two-grid compact difference (TGCD) scheme for solving the viscous Burgers' equation. The two-grid technique has received a lot of attention since it was initially developed in [38, 39] due to its high efficiency. It has been widely used in a variety of situations, such as the two-grid method coupled with mixed finite element scheme [11], the temporal two-grid scheme for nonlinear time-fractional mobile/immobile transport equation [28], the spatial two-grid method combined with mixed finite element scheme for Burgers' equation [18], etc. To the best of the authors' knowledge, for the Burgers' equation, all two-grid-based numerical techniques are, nonetheless, discrete techniques for space. The fundamental reason for using the two-grid approach is to reduce the computational cost by obtaining an approximate solution on the coarse grid and then a more accurate solution on the fine grid, where the approximate solution can serve as an estimate to make the computational cost on the fine grid as economical as possible. Therefore, it becomes a natural choice to apply the two-grid method on the temporal discretization, which, unfortunately, to the authors' knowledge, has not been developed for the Burgers' equation.

Hence, in this paper, by combining the two-grid method for temporal discretization and the compact finite difference method for spatial discretization, we arrive at a novel fully discrete scheme for solving the viscous Burgers' equation. More precisely, we use the second-order Crank-Nicolson method to discretize the time derivative, the compact finite difference formula to approximate the spatial second-order term, and the developed nonlinear fourth-order operator to handle the nonlinear convection term. The temporal two-grid approach is realized by three steps. First, on the coarse grid, to establish a nonlinear compact finite difference scheme, we use fixed point iterative method to calculate its numerical solution. Second, we apply the Lagrange interpolation formula to obtain the rough numerical solution on the fine grid. Third, on the fine grid, to formulate linear compact finite difference scheme, based on the results of the first two steps, then we can obtain numerical solution on the fine grid. By such a strategy, a numerical scheme that is not only second-order accurate in time and fourth-order accurate in space,