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A Fast Preconditioning Strategy for QSC-CN Scheme of Space Fractional Diffusion Equations and Its Spectral Analysis

Wei Qu[1], Yuanyuan Huang[2] and Siu-Long Lei[2,*]

[1] *School of Mathematics and Statistics, Shaoguan University, Shaoguan, Guangdong 512005, China*
[2] *Department of Mathematics, University of Macau, Macau SAR, China*

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Abstract. A quadratic spline collocation method combined with the Crank-Nicolson time discretization of the space fractional diffusion equations gives discrete linear systems, whose coefficient matrix is the sum of a tridiagonal matrix and two diagonal-multiply-Toeplitz-like matrices. By exploiting the Toeplitz-like structure, we split the Toeplitz-like matrix as the sum of a Toeplitz matrix and a rank-2 matrix and Strang's circulant preconditioner is constructed to accelerate the convergence of Krylov subspace method like generalized minimal residual method for solving the discrete linear systems. In theory, both the invertibility of the proposed preconditioner and the clustering spectrum of the corresponding preconditioned matrix are discussed in detail. Finally, numerical results are given to demonstrate that the performance of the proposed preconditioner is better than that of the generalized T. Chan's circulant preconditioner proposed recently by Liu et al. (J. Comput. Appl. Math., 360 (2019), pp. 138–156) for solving the discrete linear systems of one-dimensional and two-dimensional space fractional diffusion equations.

AMS subject classifications: 26A33, 65L12, 65L20, 65N15

Key words: Circulant preconditioning, Toeplitz-like matrix, matrix splitting, spectral analysis, Krylov subspace iterative method.

1 Introduction

In this paper, we consider a fast preconditioning strategy for the following initial-boundary value problem of one-dimensional (1D) space fractional diffusion equations

*Corresponding author.
Emails: quwei@sgu.edu.cn (W. Qu), yc07487@umac.mo (Y. Huang), sllei@um.edu.mo (S-L. Lei)

(SFDEs) with variable coefficients:

$$
\begin{cases}
\dfrac{\partial u(x,t)}{\partial t} = k_+(x,t)\dfrac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} + k_-(x,t)\dfrac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} + f(x,t), & (x,t)\in\Omega\times(0,T], \\
u(x,t)=0, & (x,t)\in(\mathbb{R}\backslash\Omega)\times(0,T], \\
u(x,0)=u_0(x), & x\in\overline{\Omega},
\end{cases}
\tag{1.1}
$$

where $\alpha\in(1,2)$ is the order of the fractional derivative, $\Omega=(x_L,x_R)$, $\partial\Omega$ denotes its boundary, $\overline{\Omega}=\Omega\cup\partial\Omega$, $f(x,t)$ is the source term, $k_\pm(x,t)$ are two nonnegative bounded diffusion coefficient functions, $u_0(x)$ is the initial condition, and $\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha}$ and $\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha}$ denote the left- and right-sided Riemann-Liouville (R-L) fractional derivatives, whose definitions are given by [29]:

$$
\begin{aligned}
\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} &= \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_{x_L}^{x}\frac{u(\xi,t)}{(x-\xi)^{\alpha-1}}d\xi, \\
\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} &= \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_{x}^{x_R}\frac{u(\xi,t)}{(\xi-x)^{\alpha-1}}d\xi,
\end{aligned}
$$

in which $\Gamma(\cdot)$ denotes the Gamma function. We remark that, as mentioned in [10], we impose the so-called absorbing boundary conditions, and assume that the solution is zero on $\mathbb{R}\backslash\Omega$ at each time level in Eq. (1.1).

During the last few decades, fractional differential equations have received mainstream concern in various fields, such as image processing [1], physics [4,5], biology [27], and finance [34,43]. Indeed, the development of seeking the analytical solutions of fractional differential equations is not trivial at all; even though there are very few cases of fractional differential equations in which the closed-form analytical solutions are known. Therefore, research on numerical methods for fractional differential equations is meaningful and has attracted enormous attention. Along this line of research, finite difference method (FDM) [16,29,30,35,36,38,40,44], as one of the most popular methods, has been widely used in the discretization of problems in the form (1.1).

In the derivation of FDM, the nonlocal features of fractional derivatives often produce a large and full discrete linear system, which requires $\mathcal{O}(N^3)$ complexity and $\mathcal{O}(N^2)$ storage by using Gaussian elimination, where N refers to the matrix size. Fortunately, a special Toeplitz-like coefficient matrix with the property of strictly diagonally dominant was found by Wang and his colleagues, and they proposed a direct FDM with the computational cost of $\mathcal{O}(N\log^2 N)$ operations to solve the corresponding resulting linear systems, see [41,42]. After that, many fast numerical methods are subsequently developed for these Toeplitz-like linear systems, which can significantly reduce the computational complexity and the memory requirement. Among them, we mention some fast numerical methods like the multigrid method [11,32], circulant preconditioning technique [15], circulant-based approximate inverse preconditioner method [31], banded preconditioner method [17], structure preserving preconditioner method [12], limited memory block preconditioners method [7], matrix splitting iteration method [2,25,37], and so on.