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Error Analysis of a New Euler Semi-Implicit Time-Discrete Scheme for the Incompressible MHD System with Variable Density

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Abstract. The incompressible magnetohydrodynamics system with variable density is coupled by the incompressible Navier-Stokes equations with variable density and the Maxwell equations. In this paper, we study a new first-order Euler semi-discrete scheme for solving this system. The proposed numerical scheme is unconditionally stable for any time step size $\tau > 0$. Furthermore, a rigorous error analysis is presented and the first-order temporal convergence rate $\mathcal{O}(\tau)$ is derived by using the method of mathematical induction and the discrete maximal L^p-regularity of the Stokes problem. Finally, numerical results are given to support the theoretical analysis.

AMS subject classifications: 65N30, 76M05

Key words: Magnetohydrodynamics, variable density flows, Euler semi-implicit scheme, error analysis.

1 Introduction

In this paper, we consider the 3D incompressible magnetohydrodynamics (MHD) system with variable density. It is used to describe the motions of several conducting incompressible immiscible fluids without surface tension in presence of a magnetic field, and has a wide range of applications in physical and industrial fields, such as astrophysics, geophysics, plasma physics and liquid metals of an aluminum electrolysis cell (cf. [8, 16, 27, 39]). This MHD system is governed by the following nonlinear parabolic system in $Q_T = \Omega \times (0,T]$:

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1.1a}$$

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$$\rho(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u})-\frac{1}{Re}\Delta\mathbf{u}+S\mathbf{b}\times\mathbf{curl}\ \mathbf{b}+\nabla p=\mathbf{f}, \tag{1.1b}$$

$$\mathbf{b}_t+\frac{1}{Rm}\mathbf{curl}\ (\mathbf{curl}\ \mathbf{b})-\mathbf{curl}\ (\mathbf{u}\times\mathbf{b})=0, \tag{1.1c}$$

$$\nabla\cdot\mathbf{u}=0,\quad \nabla\cdot\mathbf{b}=0, \tag{1.1d}$$

where $\Omega\subset\mathbf{R}^3$ is a bounded and convex domain with the boundary $\Gamma=\partial\Omega$ and $[0,T]$ is the time interval with some $T>0$. The vector function $\mathbf{f}$ is a given body force. Positive constants Re, Rm, S represent the Reynolds number, the magnetic Reynolds number and the coupling number, respectively. The unknown functions are the density ρ, the velocity field $\mathbf{u}$, the pressure p and the magnetic field $\mathbf{b}$.

The MHD system (1.1a)-(1.1d) is supplemented with the following initial values and boundary conditions:

$$\begin{cases} \rho(x,0)=\rho_0(x), & \mathbf{u}(x,0)=\mathbf{u}_0(x), & \mathbf{b}(x,0)=\mathbf{b}_0(x) & \text{in } \Omega, \\ \mathbf{u}(x,t)=\mathbf{g}(x,t), & \mathbf{b}(x,t)\cdot\mathbf{n}=0, & \mathbf{curl}\ \mathbf{b}(x,t)\times\mathbf{n}=0 & \text{on } \Sigma_T, \\ \rho(x,t)=a(x,t) & & & \text{on } \Sigma_T^{in}, \end{cases} \tag{1.2}$$

where $\mathbf{n}$ denotes the outward unit normal vector to the boundary Γ, $\Sigma_T=\Gamma\times(0,T]$, $\Sigma_T^{in}=\Gamma_{in}\times(0,T]$ and Γ_{in} is the inflow boundary defined by $\Gamma_{in}=\{x\in\Gamma:\ \mathbf{g}\cdot\mathbf{n}<0\}$. For the reason of simplicity, we consider the homogeneous Dirichlet boundary condition for the velocity field, i.e., $\mathbf{g}=0$. This means that the boundary is impermeable, i.e., $\Gamma_{in}=\emptyset$.

In addition, we require that the initial velocity field $\mathbf{u}_0$ and magnetic field $\mathbf{b}_0$ satisfy the incompressible conditions, i.e.,

$$\nabla\cdot\mathbf{u}_0=\nabla\cdot\mathbf{b}_0=0 \tag{1.3}$$

and there is no vacuum state in Ω, i.e.,

$$0<\min_{x\in\overline{\Omega}}\rho_0(x):=m\le\rho_0(x)\le M:=\max_{x\in\overline{\Omega}}\rho_0(x). \tag{1.4}$$

There are many studies on the well-posedness of the solutions to the MHD system (1.1a)-(1.2). The global existence of weak solutions of finite energy in the whole space $\mathbf{R}^3$ was firstly established by Gerbeau and Le Bris under no vacuum assumption (1.4) in [15]. The global existence of strong solutions with small initial data in some Besov space was proved by Abidi and Paicu in [1]. Chen, Tan and Wang in [11] considered the local strong solutions in the presence of vacuum. Huang and Wang in [26] extended it to the global existence of strong solutions for 2D problem. Other theoretical results can be found in [7, 12, 23, 34] and references cited therein.

On numerical methods for the incompressible MHD problems, there have a large amount of works for the constant density MHD problem, such as [5, 6, 14, 24, 33, 37, 38, 40, 42–46] and references cited therein. However, few studies are made for the variable