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Finite Volume Element Method for a Nonlinear Parabolic Equation

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Abstract. In this article, we study a parabolic equation with both nonlinear time-derivative term and nonlinear diffusion term by the finite volume element method. The optimal error estimate in H^1-norm is proved for fully discrete scheme. The sub-optimal error estimate in L^2-norm is proved both for semi-discrete scheme and fully discrete scheme. We prove the existence of solution for the fully discrete scheme. Numerical results show the effectiveness of our method.

AMS subject classifications: 65N08, 65N15

Key words: Nonlinear parabolic equation, error estimate, finite volume element method.

1 Introduction

The theory of numerical methods for linear equations is relatively mature. In contrast, the research findings of nonlinear equations are inadequate. Based on the different positions of nonlinear terms, nonlinear equations are divided into different types. For the nonlinear parabolic equation, a common case is that the diffusion term or the convective term is nonlinear, and many authors have done a lot of work in this area. However, only a few researchers pay attention to the situation that the time derivative is nonlinear. In fact, this type of equations are also of great importance in describing phenomenons in natural and social science. For the radiation diffusion equation [28], the energy can be a function of the temperature in some cases. In this condition, the radiation diffusion system can be transformed into a equation with nonlinear time derivative. In [1], the authors used an equation with nonlinear time derivative to simulate the flow in root-soil system. In

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this paper. we consider the following nonlinear parabolic equation with both nonlinear diffusion term and nonlinear time derivative term:

$$\begin{cases} \dfrac{\partial a(u)}{\partial t} - \nabla\cdot(b(u))\nabla u) = f(\mathbf{x},t), & (\mathbf{x},t)\in\Omega\times I, \\ u(\mathbf{x},t) = g(\mathbf{x},t), & (\mathbf{x},t)\in\partial\Omega\times I, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), & \mathbf{x}\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset R^2$ is a bounded domain and $I=(0,T]$ is the time interval. We assume that there exists constants α_i, $i=1,2,3,4$ and β_i, $i=1,2,3$, satisfying

$$0<\alpha_1\le a(u)\le\alpha_2, \quad 0<\alpha_3\le a'(u)\le\alpha_4, \quad |a''(u)|\le\alpha_5, \tag{1.2a}$$

$$0<\beta_1\le b(u)\le\beta_2, \quad |b'(u)|\le\beta_3. \tag{1.2b}$$

For the elliptic equation with nonlinear diffusion term, Douglas and Dupont [14] proposed a Galerkin method for the nonlinear Dirichlet problem in 1974. Later, Liu [19] et al. also used finite element method (FEM) to solve the nonlinear elliptic equation, and extended the coefficient matrix to a more general case. Due to the property of local conservation, finite volume element method [11, 16, 20, 27] (FVEM) is attracting more and more attention. In terms of FVEM, Li [18] gave a linear element finite volume method for this equation and obtained the error estimate in H^1-norm. Chatzipantelidis [5] et al. provided a new proof and got the optimal error estimate both in H^1-norm and L^2-norm. Bi [3] et al. discussed a two-grid finite volume element method for nonlinear elliptic equation. Recently, Du [15] et al. discussed a quadratic FVEM for the nonlinear elliptic equation and established the optimal error estimates.

For the parabolic equation with nonlinear diffusion term and linear time derivative, there are lots of articles both in FEM and FVEM. For the FEM, Douglas and Dupont [13] discussed some linear and nonlinear parabolic equations with Galerkin method and obtained optimal H^1-norm error estimate in 1970. On this basis, Wheeler [21] used Galerkin method to solve a nonlinear parabolic problem and derived L^2-norm error estimate. Chen [9] et al. analyzed a two-grid expanded mixed finite element method for the nonlinear parabolic equation. Yang [23] used the least-squares mixed finite element to solve the nonlinear convection-diffusion equation. Cannon and Lin [4] studied finite element method for the nonlinear diffusion equation with memory and gave error estimate in L^2-norm. For the FVEM, Wu [22] solved a nonlinear parabolic equation by the generalized difference method and obtained the optimal H^1-norm error estimate in 1987. Chatzipantelidis and Ginting [6] studied a nonlinear parabolic equation by the finite volume element method and got the error estimate under a mild mesh condition. Chen and Liu [8] used the finite volume element method solving a nonlinear parabolic problem, where the diffusion term is nonlinear. Zhang [25] presented a semi-discrete finite volume element scheme to solve a nonlinear parabolic equation, where the convection term, diffusion term and reaction term are all nonlinear. Zhang [26] et al. gave a full-discrete two-grid finite volume element scheme to solve the nonlinear parabolic equation. Yang and