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Error Analysis of Fully Discrete Data Assimilation Algorithms for Reaction-Diffusion Equation

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Abstract. In this paper we propose a continuous downscaling data assimilation algorithm for solving reaction-diffusion equations with a critical parameter. For the spatial discretization we consider the finite element methods. Two backward differentiation formulae (BDF), a backward Euler method and a two-step backward differentiation formula, are employed for the time discretization. Employing the dissipativity property of the underlying reaction-diffusion equation, under suitable conditions on the relaxation (nudging) parameter and the critical parameter, we obtain uniform-in-time error estimates for all the methods for the error between the fully discrete approximation and the reference solution corresponding to the measurements given on a coarse mesh by an interpolation operator. Numerical experiments verify and complement our theoretical results.

AMS subject classifications: 34D06, 65M12, 35K57, 35Q93, 37C50

Key words: Data assimilation, reaction-diffusion equation, finite element method, BDF methods, fully discrete, uniform-in-time error estimates, Allen-Cahn equation.

1 Introduction

In this paper we study the data assimilation approximation of reaction-diffusion equations with a critical parameter ϵ,

$$\partial_t u - \Delta u + \frac{1}{\epsilon^2} f(u) = g, \qquad (x,t) \in \Omega \times (0,T], \tag{1.1a}$$

$$u = 0, \qquad (x,t) \in \partial\Omega \times (0,T], \tag{1.1b}$$

based on the fully discrete backward differentiation formula (BDF) finite element (FE) methods, where $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, is a bounded domain with $\partial\Omega$ being either smooth or a convex polygon, the critical parameter $\epsilon > 0$ is a phenomenological constant modeling

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the effect of interfacial energy, g is a smooth function, and f is the derivative of a smooth function. A typical example of f is

$$f(u):=F'(u) \quad \text{and} \quad F(u)=\frac{1}{4}\left(u^2-1\right)^2.$$

The differential equation (1.1) arises in models of chemical reactions and in mathematical biology (see, e.g., [28,46]). When $g\equiv 0$, Eq. (1.1) is the well-known Allen-Cahn equation, which describes the phase separation in binary alloy systems, complex fluids and soft matter; see [2,11,13,17,18,52,54] and the references therein.

Much attention has been devoted to the numerical solutions of the reaction-diffusion equation (1.1) with an available initial condition for u (see, e.g., [18,19,29,33–35,40,45]). However, in performing accurate, practical simulation of this system, one typically does not have complete information about the initial data which may not be measured accurately, or simply unknown. How to obtain an accurate numerical solution in this case becomes a formidable challenge in practical engineering simulation. To address this issue, following the idea of [6], we consider instead a solution v of the following system

$$\partial_t v-\Delta v+\frac{1}{\epsilon^2}f(v)=g+\beta(I_H(u)-I_H(v)), \quad (x,t)\in\Omega\times(0,T], \tag{1.2}$$

which is referred to as the data assimilation equation. Here β is the nudging parameter, I_H denotes an operator used for interpolating the coarse spatial mesh measurements, corresponding to a solution u of (1.1), observed at a coarse spatial mesh, and H denotes the resolution of the coarse spatial mesh. The measurements are assumed to be continuous in time and error-free.

Since data assimilation can obtain better predictions in a physical system by combining a forecast model with observational data, there is a vast literature on data assimilation methods (see, e.g., [1,4,5,7,12,23,24,30,32,36,37,39,41,43,47]). Recently, continuous data assimilation [6] has been introduced for a large class of dissipative partial differential equations, including Rayleigh-Bénard convection [15], the planetary geostrophic ocean dynamics model [16], the Chafee-Infante reaction-diffusion equation, the Kuramoto-Sivashinsky equation [37], Rayleigh-Bénard convection equations [3,14], and the Navier-Stokes equations [22, 27], etc.. Many studies have been conducted on the error estimates of semi-discrete finite element data assimilation algorithms (see, e.g., [9, 21, 38]). However, little research has been performed on the numerical analysis of fully discrete finite element data assimilation algorithms. To the best of our knowledge, there are only [20, 31, 42, 53] focused on the numerical analysis of fully discrete data assimilation methods for the Navier-Stokes equations and the fluid transport equations. In this paper, we will carry out the numerical analysis of the fully discrete continuous data assimilation method for the reaction-diffusion equation based on finite element methods in space and BDF methods in time. Different from the Navier-Stokes equation, the critical parameter ϵ is coupled with the nonlinear term f.