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An Energy Stable Local Discontinuous Galerkin Method for a Binary Compressible Flow

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Abstract. This paper focuses on an energy-stable local discontinuous Galerkin (LDG) method for a binary compressible flow model. Since the densities and the momentum are highly coupled in the equations, and the test and basis functions in LDG discretizations have to be in the same finite element space, it is difficult to obtain stable LDG discretizations for the binary compressible flow model. To tackle this issue, we take the mass average velocity **v** and its square as auxiliary variables. These auxiliary variables are chosen in the stability analysis as the test functions for the momentum and density balance equations, respectively. Using the Crank-Nicolson (CN) time integration method, we can prove then the stability of the LDG-CN discretization. Computations are provided to demonstrate the accuracy, efficiency and capabilities of the numerical method.

AMS subject classifications: 65M15, 65M60

Key words: Binary compressible flow, energy-stable, local discontinuous Galerkin method, Crank-Nicolson time integration method

1 Introduction

Fluid mixtures exist in nature as well as in industry. Some fluid mixtures are compressible, for example, the binary flows of non-hydrocarbon (e.g., CO_2) and hydrocarbons in the enhanced oil recovery process [15,21,34]. The model we focus on is a thermodynamically consistent hydrodynamic model for compressible binary fluid mixtures [17,18,22,32,37,38]. In this paper, we study a local discontinuous Galerkin (LDG) method for the numerical solution of the binary compressible flow.

We consider a compressible binary fluid in a domain $\Omega \in \mathbb{R}^d$ with $d=2$. Let ρ_i be the mass density of each compressible component in the fluid mixture, and **v** the mass average velocity. The governing equations of the compressible binary fluid, in dimensionless

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form, read

$$\frac{\partial \rho_1}{\partial t}+\nabla\cdot(\rho_1\mathbf{v})=\nabla\cdot\mathbf{M}\cdot\nabla(\mu_1-\mu_2), \tag{1.1a}$$

$$\frac{\partial \rho_2}{\partial t}+\nabla\cdot(\rho_2\mathbf{v})=-\nabla\cdot\mathbf{M}\cdot\nabla(\mu_1-\mu_2), \tag{1.1b}$$

$$\frac{\partial \rho\mathbf{v}}{\partial t}+\nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v})=2\nabla\cdot\left(\frac{1}{Re_s}\mathbf{D}\right)+\nabla\left(\frac{1}{Re_v}\nabla\cdot\mathbf{v}\right)-\rho_1\nabla\mu_1-\rho_2\nabla\mu_2, \tag{1.1c}$$

in $\Omega\times(0,T]$, with $\rho=\rho_1+\rho_2$, and $\mathbf{D}=\frac{1}{2}(\nabla\mathbf{v}+\nabla\mathbf{v}^T)$. The parameters Re_s and Re_v are the Reynolds numbers. The variables μ_1,μ_2 are the chemical potentials with respect to ρ_1 and ρ_2, given by

$$\mu_1=\frac{\partial H}{\partial \rho_1}-\kappa_{11}\Delta\rho_1-\kappa_{12}\Delta\rho_2,\quad \mu_2=\frac{\partial H}{\partial \rho_2}-\kappa_{21}\Delta\rho_1-\kappa_{22}\Delta\rho_2. \tag{1.2}$$

Function $H(\rho_1,\rho_2,T)$ is the homogeneous or the bulk free energy density, which is specific to the binary fluid system studied. For example, the Flory-Huggins type free energy density function is widely used in polymeric binary fluid mixtures, given by

$$H(\rho_1,\rho_2,T)=\frac{k_BT}{m}\rho\left(\frac{1}{N_1}\frac{\rho_1}{\rho}\ln\frac{\rho_1}{\rho}+\frac{1}{N_2}\frac{\rho_2}{\rho}\ln\frac{\rho_2}{\rho}+\chi\frac{\rho_1\rho_2}{\rho^2}\right), \tag{1.3}$$

where k_B is the Boltzmann constant, T is the absolute temperature and m is the average mass of a molecule. The parameters κ_{ij}, $i,j=1,2$ measure the strength of the conformational entropy and are also assumed constants. The matrix $\mathbf{M}=(M_{ij})_{2\times2}$ is the symmetric positive mobility matrix and satisfies $\mathbf{M}\cdot\mathbf{1}=\mathbf{0}$ with $\mathbf{1}^T=(1,1)$. The initial conditions for the density and momentum are given by

$$\rho_1(\mathbf{x},t=0)=\rho_1^0(\mathbf{x}),\quad \rho_2(\mathbf{x},t=0)=\rho_2^0(\mathbf{x}),\quad \rho\mathbf{v}(\mathbf{x},t=0)=(\rho_1^0(\mathbf{x})+\rho_2^0(\mathbf{x}))\mathbf{v}^0(\mathbf{x}). \tag{1.4}$$

Periodic boundary conditions are studied in this paper. We refer to [37] for more details of the compressible binary fluid model (1.1).

To devise an energy-stable as well as high order discretization for the binary compressible fluid model (1.1)-(1.2) is non-trivial. First of all, the free energy in chemical potentials and the transport part of the equations are highly nonlinear, which can induce instabilities in the standard numerical solution. Second, higher order derivatives such as fourth order derivatives of the density make the classical numerical methods including finite difference and finite volume methods hard to extend to high order schemes [19]. Only few authors have considered the numerical solution of the binary compressible hydrodynamic equations. For example, Kou et al. [17,18], Zhao et al. [37] and Balashov [1] considered a cell-centered finite difference method for the mass balance equation and a finite volume method on the staggered mesh for the momentum balance equation. Numerical experiments show the capability of the numerical methods. In this paper, we use