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Optimal Error Analysis of Euler and BDF2 Semi-Renormalized FEMs for the Landau-Lifshitz-Slonczewski Equation

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Abstract. The Landau-Lifshitz-Slonczewski equation is used to describe the magnetization dynamics under the influence of a spin-polarized current in terms of a spin-velocity field in ferromagnetic materials. This equation is a strongly nonlinear evolution equation with a non-convex constraint of unit sphere. A natural method of preserving the non-convex constraint is the renormalized method. In this paper, we consider linearized backward Euler and BDF2 semi-renormalized finite element methods, where the renormalized numerical solution is not used in the discretizations of the time derivative and only used in linearized explicit parts. By using the piecewise linear finite element to make the spatial discretization, optimal L^2 error estimates are derived, i.e., $\mathcal{O}(\tau+h^2)$ for the Euler scheme and $\mathcal{O}(\tau^2+h^2)$ for the BDF2 scheme under some CFL type conditions. Finally, numerical results are shown to support the theoretical convergence rates.

AMS subject classifications: 65N12, 65N30, 35K61

Key words: Landau-Lifshitz-Slonczewski equation, renormalized method, finite element method, optimal error estimates.

1 Introduction

The Landau-Lifshitz (LL) equation is used to describe the dynamics of the magnetic distribution in a ferromagnetic material (cf. [24]), which reads as

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{m}\times\Delta\mathbf{m} - \alpha\mathbf{m}\times(\mathbf{m}\times\Delta\mathbf{m}) \quad \text{in } Q_T := (0,T)\times\Omega, \tag{1.1}$$

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where $T>0$ and Ω is a bounded and convex domain in $\mathbf{R}^3$. The unknown function $\mathbf{m}$ denotes the magnetization and $\alpha>0$ is the dimensionless damping parameter. It is clear that $|\mathbf{m}|$ remains unchanged in time.

It has been known that when a spin-polarized current appears, it interacts with ferromagnetic structures by the spin-transfer torque. This effect can be used to excite magnetization oscillations or to switch magnetization orientation, and it can lead to new trends for storage technologies and spintronic applications. Usually, one uses the Landau-Lifshitz-Slonczewski (LLS) equation to denote the continuum equation for the magnetization in the presence of a spin-polarized current in terms of spin-velocity fields $\mathbf{v}_1,\mathbf{v}_2:(0,T)\times\Omega\rightarrow\mathbf{R}^3$, which is a modified form of the LL equation (1.1) and written as (cf. [33,34,37])

$$\frac{\partial\mathbf{m}}{\partial t}+\mathbf{v}_1\cdot\nabla\mathbf{m}+\mathbf{m}\times(\mathbf{v}_2\cdot\nabla)\mathbf{m}=-\mathbf{m}\times\Delta\mathbf{m}-\alpha\mathbf{m}\times(\mathbf{m}\times\Delta\mathbf{m})\quad\text{in } Q_T,\tag{1.2}$$

where the spin-transfer terms $\mathbf{v}_1\cdot\nabla\mathbf{m}$ and $\mathbf{m}\times(\mathbf{v}_2\cdot\nabla)\mathbf{m}$ can be viewed as homogenized versions of Slonczewski's spin-transfer torque [22,33]. Moreover, these terms refer to fundamental work by Slonczewski [31] and Berger [9] on spin transfer between two homogeneously magnetized layers separated by paramagnetic spacers within a five-layer system.

Like as the LL equation, the magnetization $\mathbf{m}$ to the LLS equation (1.2) also satisfies the non-convex constraint $\mathbf{m}\in\mathbb{S}^2$, where $\mathbb{S}^2$ is the unit sphere in $\mathbf{R}^3$, i.e.,

$$|\mathbf{m}(t,x)|=1\quad\text{for any }\ (t,x)\in Q_T.\tag{1.3}$$

To ensure the well-posedness of the solution to the LLS equation, the appropriate initial and boundary conditions are needed. Here, we supplement

$$\mathbf{m}(0,x)=\mathbf{m}_0(x)\in\mathbb{S}^2\quad\text{in }\ \Omega\tag{1.4}$$

and

$$\frac{\partial\mathbf{m}}{\partial\mathbf{n}}=\mathbf{0}\quad\text{on }\ (0,T)\times\partial\Omega,\tag{1.5}$$

where $\mathbf{n}$ is the unit outward normal vector to the boundary $\partial\Omega$.

On the other hand, in terms of the constraint (1.3) and the following vector formula:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}\quad\text{for }\ \mathbf{a},\mathbf{b},\mathbf{c}\in\mathbf{R}^3,$$

the LLS equation (1.2) can be rewritten to an equivalent form:

$$\mathbf{m}_t-\alpha\Delta\mathbf{m}+\mathbf{m}\times\Delta\mathbf{m}+\mathbf{v}_1\cdot\nabla\mathbf{m}+\mathbf{m}\times(\mathbf{v}_2\cdot\nabla)\mathbf{m}=\alpha|\nabla\mathbf{m}|^2\mathbf{m}\quad\text{in }\ Q_T.\tag{1.6}$$

The existence of global weak solution to LLS problem has been proved in [27]. Moreover, the authors in [27] derived the existence and uniqueness of global strong solution under smallness conditions of initial data.