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The Formulation of Finite Difference Hermite RBF-WENO Schemes for Hyperbolic Conservation Laws: An Alternative Technique

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Abstract. A class of finite difference Hermite radial basis functions weighted essentially non-oscillatory (HRWENO) methods for solving conservation laws was presented by Abedian (Int. J. Numer. Meth. Fluids, 94 (2022), pp. 583–607). To reconstruct the fluxes in HRWENO, the common practice of reconstructing the flux functions was employed. In this follow-up research work, an alternative formulation to reconstruct the numerical fluxes is considered. First, the solution and its derivatives are directly employed to interpolate point values at interfaces of computational cells. Afterwards, the point values at interface of cell in building block are considered to obtain numerical fluxes. In this framework, arbitrary monotone fluxes can be employed, while in HRWENO the classical practice of reconstructing flux functions can be considered only to smooth flux splitting. Also, in the process of reconstruction these type of schemes consider the effectively narrower stencil of HRWENO methods. Extensive test cases such as Euler equations of compressible gas dynamics are considered to show the good performance of the methods.

AMS subject classifications: 65M06, 35L65

Key words: Weighted essentially non-oscillatory scheme, Hermite radial basis function interpolation, finite difference method, hyperbolic conservation laws, Euler equations.

1 Introduction

In recent decades, hyperbolic systems of conservation laws have come to the attention of researchers. Based on this, high accuracy and high resolution methods are designed to approximate the solutions of this type of equations. Among these methods, we can mention WENO and Hermite WENO (HWENO) methods. The first version of the HWENO scheme, which was in the framework of finite volume, was introduced by the authors

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of [19] to solve one-dimensional conservation laws. The scheme was extended to two dimensional cases in [20]. Due to the success of these methods, they have also been used to solve the Hamilton-Jacobi equations [21] and computational acoustics [3]. The basis of the formation of HWENO schemes is the ideas contained in WENO schemes [2,12,22, 23,28,35,36]. Today, WENO schemes are one of the most popular methods for numerically solving hyperbolic partial differential equations. The first version of the WENO schemes, which is the third-order of accuracy in a finite volume framework, was introduced by the authors of [18] in one space dimension. Then in 1996, Jiang and Shu [11] developed WENO schemes in the framework of finite differences. The advantages of this type of formulation include the design of arbitrary accurate order and expansion to multidimensional spaces.

The traditional WENO schemes, despite their vast ability to solve many practical problems, but also have disadvantages, the most important of which is the width of their numerical stencil. Moreover, a wide stencil is not optimum either in terms of an accurate treatment of weak fluctuations, or concerning the imposition of boundary conditions. Given these disadvantages and limitations, one of the best ways to manage these limitations is to use more information of the numerical solution in the neighbourhood of any given cell. HWENO and HRWENO schemes, employing the Hermite interpolation, following the classical WENO methodology, the solution and its first derivative are evolved in time and considered into the polynomial/non-polynomial reconstruction. Accordingly, HWENO/HRWENO schemes use a small stencil, but achieve the higher accuracy. HWENO/HRWENO schemes also have a disadvantage, for the same number of grid points, they require more CPU than the original WENO schemes because of the auxiliary variables which were considered.

In 1989, Shu and Osher [30] proposed a procedure to reconstruct flux functions that is easy and clean to implement. Accordingly, in most high order finite difference WENO schemes such as high order finite difference ENO [30], WENO [11], RBFWENO [7], HWENO [16] and HRWENO [1], the process of reconstruction of fluxes is based on the common practice of reconstructing the flux functions which was based on the procedure appeared in [30]. Since the non-linear stability and upwind are essential in numerical schemes, the numerical flux with finite difference in flux formulation has certain limitations in mathematical forms satisfying the forms of flux splitting. Authors of [29] proposed the alternative approach for constructing numerical fluxes in high order conservative finite difference schemes. The basis for the formation of this approach is that the ENO interpolation of the solution and its derivatives are employed to directly form the numerical flux which the interpolators are obtained directly using point values of the solution instead of flux values. The combination of WENO schemes with the basis of the approach presented in [29] with Lax-Wendroff time discretization for solving hyperbolic conservation laws has been done for the first time in [12], which overcome the defects the above stated the traditional method, like in [1,16]. Comparing the computational time in implementing WENO schemes with this approach and the standard one, it can be concluded that this approach is more expensive, but it has advantages, the most important