Analysis in Theory and Applications
Anal. Theory Appl., Vol. **41**, No. 3 (2025), pp. 197-207

DOI: 10.4208/ata.OA-2023-0009
September 2025

Estimates for Parabolic Schrödinger Operators with Certain Nonnegative Potentials

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Received 22 March 2023; Accepted (in revised version) 26 March 2025

Abstract. In this paper, the parabolic Schrödinger operator $\mathcal{P} = \partial_t - \Delta + V(x)$ on $\mathbb{R}^{n+1}$ is considered, where $n \geq 3$, nonnegative potential V belongs to the reverse Hölder class RH_q with $q \geq n/2$. The L^p boundedness of operators $V^\alpha \mathcal{P}^{-\beta}$, $V^\alpha \nabla \mathcal{P}^{-\beta}$ and their adjoint operators are established.

Key Words: L^p estimate, parabolic Schrödinger operator, reverse Hölder class.

AMS Subject Classifications: 42B25, 35J10, 42B37

1 Introduction and results

For $1 < q \leq \infty$, a nonnegative locally L^q -integrable function V is said to belong to the reverse Hölder class RH_q if there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(y)^q dy\right)^{1/q} \leq \frac{C}{|B|}\int_B V(y)dy$$

holds for every ball $B \subset \mathbb{R}^n$.

Clearly, if V belongs to RH_q with $q > 1$, then V is a Muckenhoup A_∞ weight; see [4]. From the weight theory, we know $V(x)dx$ is a doubling measure and the class RH_q has self-improvement property [5]; that is, if $V \in RH_q$ for some $q > 1$, then there exists $\epsilon > 0$ such that $V \in RH_{q+\epsilon}$.

In this paper, we consider the parabolic Schrödinger operators

$$\mathcal{P} = \partial_t - \Delta + V(x) \quad \text{on } \mathbb{R}^{n+1}, \quad n \geq 3,$$

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where the potential V belongs to the reverse Hölder class RH_q for $q \geq n/2$. We concentrate on the L^p boundedness of the operators

$$T_1 = V^\alpha \mathcal{P}^{-\beta}, \qquad 0 \leq \alpha \leq \beta \leq 1,$$
$$T_2 = V^\alpha \nabla \mathcal{P}^{-\beta}, \qquad 0 \leq \alpha \leq 1/2 \leq \beta \leq 1, \quad \beta - \alpha \geq 1/2,$$

and their adjoint operators.

Let $V \in RH_q$ for $q \geq n/2$. When $(\alpha, \beta) = (1,1)$, Gao and Jiang in [2] showed the L^p-boundedness of T_1 for $1 < p < q$. Based on this result, they obtained the L^p-boundedness of operator $\nabla^2 \mathcal{P}^{-1}$. Carbonaro et al. in [1] improved the results in [2] by the potential V with the variables $x; t$, which is essentially the generalization to $\mathbb{R}^{n+1}$ of the condition of [2]. Suppose $V \in RH_q$ with $n/2 < q < n$. When $(\alpha, \beta) = (0, 1/2)$, $1 < p < p_0$, $1/p_0 = 1/q - 1/n$, Tang and Han [7] established the L^p -boundedness of T_2; For $(\alpha, \beta) = (1/2, 1)$, $1 < p < p_0$, $1/p_0 = 3/q - 1/n$, they also obtained the the L^p -boundedness of T_2.

Assume $V \in RH_q$ with $q \geq n/2$ and $\mathcal{L} = -\Delta + V$. In [5], Shen studied the L^p-boundedness of operators $V\mathcal{L}^{-1}, V^{1/2}\mathcal{L}^{-1/2}$ and $V^{1/2}\nabla\mathcal{L}^{-1}$. In [6], Sugano researched the L^p-boundedness of operators $V^\alpha \mathcal{L}^{-\beta}$ for $0 \leq \alpha \leq \beta \leq 1$, She also obtained the L^p-boundedness of operators $V^\alpha \nabla \mathcal{L}^{-\beta}$ for $0 \leq \alpha \leq 1/2 \leq \beta \leq 1$, $\beta - \alpha \geq 1/2$.

Inspired by the above results, we have the following results.

Theorem 1.1. *Suppose $V \in RH_q$ with some $q \geq n/2$, $0 < \alpha \leq \beta \leq 1$.*

(i) If

$$\left(\frac{q}{\alpha}\right)' \leq p_1 < \frac{n+2}{2(\beta-\alpha)} \quad \text{and} \quad \frac{1}{p_2} = \frac{1}{p_1} - \frac{2(\beta-\alpha)}{n+2},$$

then

$$\|T_1^*(f)\|_{L^{p_2}(\mathbb{R}^{n+1})} \leq C\|f\|_{L^{p_1}(\mathbb{R}^{n+1})};$$

(ii) If

$$1 < p_1 \leq \frac{1}{\frac{\alpha}{q} + \frac{2(\beta-\alpha)}{n+2}} \quad \text{and} \quad \frac{1}{p_2} = \frac{1}{p_1} - \frac{2(\beta-\alpha)}{n+2},$$

then

$$\|T_1(f)\|_{L^{p_2}(\mathbb{R}^{n+1})} \leq C\|f\|_{L^{p_1}(\mathbb{R}^{n+1})}.$$

Theorem 1.2. *Suppose $V \in RH_q$ with some $q \geq n/2$, and*

$$\begin{cases} 0 \leq \alpha \leq \dfrac{1}{2} \leq \beta \leq 1, & \text{if } q \geq n, \\ 0 \leq \alpha \leq \dfrac{1}{2} < \beta \leq 1, & \text{if } n/2 \leq q < n. \end{cases}$$