Analysis in Theory and Applications
Anal. Theory Appl., Vol. **41**, No. 3 (2025), pp. 229-237

DOI: 10.4208/ata.OA-2024-0023
September 2025

A Rigidity Result for the Schiffer Conjecture on Domain with a Hole

Yingxin Sun*

School of Mathematical Sciences, Dalian University of Technology, Dalian, Liaoning 116024, China

Received 30 July 2024; Accepted (in revised version) 27 March 2025

Abstract. Let Ω be a domain with a hole containing the origin in $\mathbb{R}^2$ and u be a solution to the problem

$$-\Delta u = \mu u \quad \text{in } \Omega, \quad \partial_\nu u = 0, \quad u = c \quad \text{on } \partial^{\pm}\Omega,$$

where $\partial^{\pm}\Omega$ represents the outer and inner boundaries of Ω, respectively, c is a constant. Let μ_k denote the kth Neumann eigenvalue of the Laplacian on Ω and Ω_h is the hole. We establish that if $\mu < \mu_8$, then Ω is an annulus.

Key Words: Schiffer conjecture, overdetermined problem, symmetry.

AMS Subject Classifications: 35J25, 35N05

1 Introduction

Pompeiu problem is a mathematical problem with a long history and great significance in mathematics. It originated from Dimitrie Pompeiu's paper [1,2] in 1929. A domain $\Omega \subseteq \mathbb{R}^N$ is said to have the Pompeiu property if $f \equiv 0$ is the only continuous function satisfying

$$\int_{\sigma(\Omega)} f(x)\mathrm{d}x = 0$$

for every rigid motion σ of Ω. In 1944, Tchakaloff [3] first pointed out that the open ball fails to have the Pompeiu property. Naturally, is the ball the unique bounded smooth simply connected domain that does not have Pompeiu property [4]? This is known as the Pompeiu problem. The issue is still open. Pompeiu problem has many implications in the field of mathematics, including partial differential equation symmetry, harmonic analysis [1,2], and so on. It is worth noting that the Pompeiu problem is related to some of the problems in mathematical physics, geophysics and inverse theory. For example,

*Corresponding author. *Email address:* `sunyingxin2023@mail.dlut.edu.cn` (Y. Sun)

in geophysics, the Pompeiu problem is related to the uniqueness of the interface [7]. In addition, the Pompeiu problem is also studied in the context of the Heisenberg group [5], which involves more general geometric and analytical problems. Recent study [6] has also explored the nature of the Pompeiu problem in the one-dimensional context, particularly when considering a limited number of disconnected intervals, which makes the problem more complex and interesting.

The Pompeiu problem is also related to the so-called Schiffer conjecture.

Schiffer conjecture. Let $\Omega \subset \mathbb{R}^N$ be a bounded regular simply-connected domain. Assume $u : \Omega \to \mathbb{R}$ is a solution to the problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \\ u = \text{const} & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where λ is a parameter and ν is the unit outer normal vector on $\partial\Omega$. Then Ω is a ball and u is radially symmetric.

The Pompeiu problem or Schiffer conjecture is one of Yau's famous list of problems [18, Problem 80]. Pompeiu problem is an abstruse mathematical problem in theory, which has not been completely solved yet. However, many people continue to study this problem and obtain many important results. For instance, Berenstein and Yang [9, 10] implied that if there exists infinitely many eigenvalues for (1.1), then Ω must be a ball. In 1980, Berenstein [13] proved that Ω is a ball if $\mu = \mu_2$ in the 2-dimensional case. In 1986, Aviles [11] verified the fact that Ω is a disk when Ω is convex and $\mu \le \mu_7$. In 1994, Willms and Gladwell [17] proved that Ω is a ball if Ω is simply connected and u has no saddle points in the interior of Ω. Further, Deng [14] obtained two important results, one is that he proved the same result as Aviles [11] without the convexity assumption, and the other is that he proved that Ω is a disk if Ω is strictly convex and centrally symmetric and $\mu < \mu_{13}$. We refer to [12, 14, 15] and their references for some related results.

The above classical results all require the domain to be simply-connected. The natural question is whether Ω is an annulus if it is not simply-connected? In this paper, we aim to give a positive answer to this question via extending one of the beautiful results of Deng [14] into non simply-connected domain.

Theorem 1.1. *Let Ω be a bounded domain with a closed hole Ω_h which contains the origin in $\mathbb{R}^2$, whose outer and inner connected boundaries $\partial^\pm\Omega$ are class $C^{2,\varepsilon}$ with $\varepsilon \in (0,1)$. If ω is a solution of*

$$\begin{cases} \Delta u + \mu u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u = c & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

for some $0 < \mu < \mu_8$, then Ω is an annulus.