

## $\delta$ -Universal Taylor Series

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**Abstract.** We present a new type of universal series, termed  $\delta$ -universal series, for which the sum of squared coefficients satisfies

$$\sum_{k=0}^{\infty} |a_k|^2 < \delta$$

for an arbitrarily small  $\delta > 0$ . We establish a version of Seleznev's theorem within this framework. To construct such  $\delta$ -universal series, we develop a variation of Mergelyan's theorem.

**Key Words:** Universal series, Seleznev's Theorem, Mergelyan's theorem, hardy space.

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## 1 Introduction

Universality, which was named by Marcinkiewicz in 1935, is a generic phenomenon in analysis. As pointed in [4], "*... not every limiting process, of course, converges, but examples have been found where processes diverge in a maximal way. Such an extreme behaviour is often linked with the phenomenon of universality.*" The first example with universal flavor is Riemann rearrangement theorem, which states that if an infinite series of real numbers is conditionally convergent, then its terms can be arranged in such a way that the new series converges to any real number.

In 1914, Fekete observed that there exists a real power series  $\sum_{k=1}^{\infty} a_k x^k$  on  $[-1, 1]$  such that for every continuous function  $h$  on  $[-1, 1]$  with  $h(0) = 0$  there exists an increasing

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sequence  $(\lambda_n)$  of positive integers such that  $\sum_{k=1}^{\lambda_n} a_k x^k$  convergence to  $h(x)$  uniformly as  $n \rightarrow \infty$ . This is the first example of universal series. In 1951, Seleznev [8] constructed the first universal complex power series, which has radius of convergence 0. For every function  $h$  continuous on  $K$  and holomorphic in the interior of  $K$ , where  $K \subset \mathbb{C} \setminus \{0\}$  is compact with connected complement, there exists a subsequence of its partial sums converging uniformly to  $h$  on  $K$ . In 1971, Chui and Parnes [3] constructed universal complex power series with radius of convergence one. In 1996, Nestoridis [7] proved that the set of such universal complex power series is huge in the sense of Baire category. For a recent generalization, please refer to K. Maronikolakis and V. Nestoridis [5]. Over 100 years mathematician have published much literature on universality, a classical yet still flourishing field. Interested readers are referred to the good comprehensive survey articles [4, 9] and [2].

To demonstrate our  $\delta$ -universal series is a TRUE enhancement of Seleznev's theorem, we restate it in a form that is easy to make the comparison. For Theorem 1.1, please refer to [4, Theorem 18], [7, Theorem 2.6], and [1, Proposition 2].

**Theorem 1.1** (Seleznev, Luh, Chui, Parnes, et al.). *Let  $\mu$  be an infinite subset of  $\mathbb{N}$ . There exists a power series  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  such that*

- (i). *The radius of convergence of the series is one.*
- (ii). *For every compact set  $K \subset \{z \in \mathbb{C} : |z| > 1\}$  with connected complement and any function  $h$  continuous on  $K$  and holomorphic in the interior of  $K$  there exists an increasing sequence  $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mu$  such that the partial sums  $S_{\lambda_n}(f)(z) = \sum_{k=0}^{\lambda_n} a_k z^k$ ,  $n = 1, 2, \dots$  satisfy*

$$\limsup_{n \rightarrow \infty} \sup_{z \in K} |S_{\lambda_n}(f)(z) - h(z)| = 0.$$

As (i),  $\sum_{k=0}^{\infty} a_k z^k$  is the Taylor development of  $f(z)$ , and  $\sum_{k=0}^{\infty} a_k z^k$  is a universal Taylor series.

Denote  $U$  by the set of power series satisfying (i) and (ii) in Theorem 1.1. V. Nestoridis proved [7, Theorem 2.6] proved the following.

**Theorem 1.2** (V. Nestoridis).  *$U$  is a  $G_\delta$ -dense subset of the space  $H(\mathbb{D})$  of holomorphic functions in the open unit disk  $\mathbb{D}$  in complex plan  $\mathbb{C}$  endowed with the topology of uniform convergence on compacta.*

The goal of this study is to develop a new type of restricted universal series, called  $\delta$ -universal series. The paper is organized as follows.

In Section 2, we define the set  $H_\delta$  where  $\delta$ -universal series reside, and state our main results. In Section 3, First, we establish a  $\delta$ -restricted Seleznev's Theorem (Lemma 3.1, beside the approximation property, the magnitude of coefficients  $\sum_{k=1}^{\infty} |a_k|^2$  is bounded by an arbitrarily small  $\delta > 0$ . Next, we prove two theorems. In Section 4, we make some comments, point out some differences between  $\delta$ -universal series and results in [1].