

δ -Universal Taylor Series

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Received 22 April 2025; Accepted (in revised version) 19 July 2025

Dedicated to the memory of Prof. Donggao Deng on the occasion of his 90th birthday

Abstract. We present a new type of universal series, termed δ -universal series, for which the sum of squared coefficients satisfies

$$\sum_{k=0}^{\infty} |a_k|^2 < \delta$$

for an arbitrarily small $\delta > 0$. We establish a version of Seleznev's theorem within this framework. To construct such δ -universal series, we develop a variation of Mergelyan's theorem.

Key Words: Universal series, Seleznev's Theorem, Mergelyan's theorem, hardy space.

AMS Subject Classifications: 30K05, 30E05, 41A58

1 Introduction

Universality, which was named by Marcinkiewicz in 1935, is a generic phenomenon in analysis. As pointed in [4], "*... not every limiting process, of course, converges, but examples have been found where processes diverge in a maximal way. Such an extreme behaviour is often linked with the phenomenon of universality.*" The first example with universal flavor is Riemann rearrangement theorem, which states that if an infinite series of real numbers is conditionally convergent, then its terms can be arranged in such a way that the new series converges to any real number.

In 1914, Fekete observed that there exists a real power series $\sum_{k=1}^{\infty} a_k x^k$ on $[-1, 1]$ such that for every continuous function h on $[-1, 1]$ with $h(0) = 0$ there exists an increasing

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sequence (λ_n) of positive integers such that $\sum_{k=1}^{\lambda_n} a_k x^k$ convergence to $h(x)$ uniformly as $n \rightarrow \infty$. This is the first example of universal series. In 1951, Seleznev [8] constructed the first universal complex power series, which has radius of convergence 0. For every function h continuous on K and holomorphic in the interior of K , where $K \subset \mathbb{C} \setminus \{0\}$ is compact with connected complement, there exists a subsequence of its partial sums converging uniformly to h on K . In 1971, Chui and Parnes [3] constructed universal complex power series with radius of convergence one. In 1996, Nestoridis [7] proved that the set of such universal complex power series is huge in the sense of Baire category. For a recent generalization, please refer to K. Maronikolakis and V. Nestoridis [5]. Over 100 years mathematician have published much literature on universality, a classical yet still flourishing field. Interested readers are referred to the good comprehensive survey articles [4, 9] and [2].

To demonstrate our δ -universal series is a TRUE enhancement of Seleznev's theorem, we restate it in a form that is easy to make the comparison. For Theorem 1.1, please refer to [4, Theorem 18], [7, Theorem 2.6], and [1, Proposition 2].

Theorem 1.1 (Seleznev, Luh, Chui, Parnes, et al.). *Let μ be an infinite subset of \mathbb{N} . There exists a power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ such that*

- (i). *The radius of convergence of the series is one.*
- (ii). *For every compact set $K \subset \{z \in \mathbb{C} : |z| > 1\}$ with connected complement and any function h continuous on K and holomorphic in the interior of K there exists an increasing sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mu$ such that the partial sums $S_{\lambda_n}(f)(z) = \sum_{k=0}^{\lambda_n} a_k z^k$, $n = 1, 2, \dots$ satisfy*

$$\limsup_{n \rightarrow \infty} \sup_{z \in K} |S_{\lambda_n}(f)(z) - h(z)| = 0.$$

As (i), $\sum_{k=0}^{\infty} a_k z^k$ is the Taylor development of $f(z)$, and $\sum_{k=0}^{\infty} a_k z^k$ is a universal Taylor series.

Denote U by the set of power series satisfying (i) and (ii) in Theorem 1.1. V. Nestoridis proved [7, Theorem 2.6] proved the following.

Theorem 1.2 (V. Nestoridis). *U is a G_δ -dense subset of the space $H(\mathbb{D})$ of holomorphic functions in the open unit disk \mathbb{D} in complex plan \mathbb{C} endowed with the topology of uniform convergence on compacta.*

The goal of this study is to develop a new type of restricted universal series, called δ -universal series. The paper is organized as follows.

In Section 2, we define the set H_δ where δ -universal series reside, and state our main results. In Section 3, First, we establish a δ -restricted Seleznev's Theorem (Lemma 3.1, beside the approximation property, the magnitude of coefficients $\sum_{k=1}^{\infty} |a_k|^2$ is bounded by an arbitrarily small $\delta > 0$. Next, we prove two theorems. In Section 4, we make some comments, point out some differences between δ -universal series and results in [1].

2 Main results

In this paper, we are interested in restricted universal series. That is, the coefficients $\{a_k\}_{k=1}^\infty$ in the series satisfy certain constrain condition.

A famous result is due to Menshov [6] and Talalyan [10]. They proved [4, Theorem 13] that for every complete orthogonal system $\{\varphi_k\}_{k=0}^\infty$ in $L^2[0, 1]$, there exists a universal series $\sum_{k=0}^\infty a_k \varphi_k$ such that $a_k \rightarrow 0$ as $k \rightarrow \infty$.

To author's knowledge, all constrains on restricted universal series are added on single coefficient. In contrast, we will enforce a much more stronger overall constrain

$$\sum_{k=1}^\infty |a_k|^2 < \delta.$$

Bayart, Grosse-Erdmann, Nestoridis and Papadimitropoulos [1] develop an abstract theory of restricted universal series. Our δ -universal series is inspired by their great work, but not been covered in their abstract settings. The main difference will be discussed in the final section.

We present two theorems corresponding to Theorems 1.1 and 1.2, respectively.

Theorem 2.1 (δ -Restricted Seleznev's theorem). *Let $0 < \delta < 1$ and μ be an infinite subset of \mathbb{N} . There exists a power series $f(z) = \sum_{k=1}^\infty a_k z^k$ such that*

- (i). *The radius of convergence of the series is one.*
- (ii). *For every compact set $K \subset \{z \in \mathbb{C} : |z| > 1\}$ with connected complement and any function h continuous on K and holomorphic in the interior of K there exists an increasing sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subset \mu$ such that*

$$\limsup_{n \rightarrow \infty} \sup_{z \in K} |S_{\lambda_n}(f)(z) - h(z)| = 0.$$

- (iii). $\sum_{k=1}^\infty |a_k|^2 < \delta$. *The series is called a δ -universal series. Denote U_δ^μ the set of δ -universal series in this theorem.*

Comparing with Theorem 1.1, (iii) is a new restriction, thus Theorem 2.1 is a TRUE refinement of Theorem 1.1.

To demonstrate the G_δ density, we introduce a subset of Hardy space $H^2(\mathbb{D})$, where δ -universal series resides. Define

$$H_\delta(\mathbb{D}) = \left\{ f \in H^2(\mathbb{D}) : f(z) = \sum_{k=1}^\infty a_k z^k, \|f\|_H = \left(\sum_{n=1}^\infty |a_n|^2 \right)^{1/2} < \delta \right\}$$

Theorem 2.2. *Let $0 < \delta < 1$ and μ be an infinite subset of \mathbb{N} . The set U_δ^μ is a G_δ dense subset of $H_\delta(\mathbb{D})$.*

Remark 2.1. We emphasize that $H_\delta(\mathbb{D})$ is an open ball of Hardy space, not a linear space, and G_δ density is established in the topology induced from $H^2(\mathbb{D})$, not the topology of $H(\mathbb{D})$.

Remark 2.2. In [7], V. Nestoridis enlarged the residence region of compact set K from $\{z \in \mathbb{C} : |z| > 1\}$ to $\{z \in \mathbb{C} : |z| \geq 1\}$ in Theorem A, and solved many boundary value problems. In theorem 2.1 (theorem 2.2 too), instead of $K \subset \{z \in \mathbb{C} : |z| > 1\}$, can we use the condition $K \subset \{z \in \mathbb{C} : |z| \geq 1\}$? The answer is NO.

Note that theorem 2.2 shows that every δ -universal series is a function in $H^2(\mathbb{D})$. On the other side, V. Nestoridis has proved that [7, Proposition 3.5] if a series $\sum_{k=0}^{\infty} a_k z^k$ satisfying (i) and (ii) in Theorem A for every compact $K \subset \{z \in \mathbb{C} : |z| \geq 1\}$, then $\sum_{k=0}^{\infty} a_k z^k$ does not belong to the space $H^2(\mathbb{D})$. Hence $K \subset \{z \in \mathbb{C} : |z| > 1\}$ is a reasonable requirement for δ -universal series.

3 Proof of main results

We first give a variant of Mergelyan's theorem, then prove Theorems 2.1 and 2.2. In this section, all coefficients of polynomial or series are complex.

Lemma 3.1 (δ -Restricted Mergelyan's theorem). *Suppose that $K \subset \{z \in \mathbb{C} : |z| > 1\}$ is a non-empty compact set having connected complement and f is a function continuous on K and holomorphic in the interior of K . Let $0 < \delta < 1$. Then for any $\varepsilon > 0$ and integer $N \geq 1$, there exists a polynomial $p(z) = \sum_{k=0}^n a_{k+N} z^{k+N}$ satisfying:*

- (i). $\sup_{z \in K} |f(z) - p(z)| < \varepsilon$.
- (ii). $\sum_{k=0}^n |a_{k+N}|^2 < \delta$.

Proof. By the assumption of K , the set $K \cup \overline{\mathbb{D}}$ is compact having connected complement. Define a function

$$F(z) = \begin{cases} f(z)/z^N, & \text{if } z \in K, \\ 0, & \text{if } z \in \overline{\mathbb{D}}. \end{cases}$$

Then F is continuous on $K \cup \overline{\mathbb{D}}$ and holomorphic in the interior of $K \cup \overline{\mathbb{D}}$.

Let $M = \sup\{|z| : z \in K\}$, then $M < \infty$, and choose $\varepsilon_1 > 0$ such that $\varepsilon_1 < \min\{\varepsilon/M^N, \delta^{1/2}\}$. Apply Mergelyan's theorem, there exists a polynomial $q(z) = \sum_{k=0}^n a_k z^k$ such that for all $z \in K \cup \overline{\mathbb{D}}$

$$|F(z) - q(z)| < \varepsilon_1.$$

Setting

$$p(z) = q(z)z^N = \sum_{k=0}^n a_k z^{k+N},$$

we have

$$\sup_{z \in K} |f(z) - p(z)| = \sup_{z \in K} |z^N| |f(z)/z^N - q(z)| < \varepsilon_1 M^N < \varepsilon. \tag{3.1}$$

Thus $p(z)$ satisfies the requirement (i).

Next we estimate $\sum_{k=0}^n |a_k|^2$. By the orthogonality of $\{z^k\}_{k=0}^n$ on the unit circle,

$$\frac{1}{2\pi} \int_0^{2\pi} |q(e^{i\theta})|^2 d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k=0}^n a_k e^{ik\theta} \right) \left(\sum_{k=0}^n \bar{a}_k e^{-ik\theta} \right) d\theta = \sum_{k=0}^n |a_k|^2.$$

The facts $F(z) = 0$ and $|F(z) - q(z)| < \varepsilon_1$ on $\bar{\mathbb{D}}$ imply $|q(z)| < \varepsilon_1$ on $\bar{\mathbb{D}}$. Thus

$$\sum_{k=0}^n |a_k|^2 = \frac{1}{2\pi} \int_0^{2\pi} |q(e^{i\theta})|^2 d\theta < \varepsilon_1^2 < \delta. \tag{3.2}$$

So $p(z)$ satisfies the requirement (ii), and the proof of Lemma 3.1 is completed. \square

Proof of Theorem 2.1. By [7, Lemma 2.1], there exist countable compact sets $K_m \subset \{z \in \mathbb{C} : |z| > 1\}$, $m = 1, 2, \dots$ with connected complements, such that every non-empty compact set $K \subset \{z \in \mathbb{C} : |z| > 1\}$ having connected complement is contained in some K_m . Let $f_j, j = 1, 2, \dots$, be an enumeration of all polynomials having coefficients with rational coordinates. The Mergelyan's theorem asserts that $\{f_j\}_{j=1}^\infty$ is dense in the space of functions that are continuous on K_m and holomorphic in the interior of K_m for all $m = 1, 2, \dots$.

Construct a list \mathcal{L} of all the pairs $(f_j; K_m), j = 1, 2, \dots, m = 1, 2, \dots$, such that every pair occurs infinitely often in \mathcal{L} . Write the list by $\mathcal{L} = \{(g_1; L_1), (g_2; L_2), \dots\}$, where g_i is in $\{f_j\}$ and L_i is in $\{K_m\}, j = 1, 2, \dots, m = 1, 2, \dots$.

The construction is performed by induction on $(g_n; L_n)$ in three steps.

Step 1. $n = 1$. By Lemma 3.1, there exists a polynomial $q_1(z) = \sum_{k=1}^{m_1} a_k z^k$ such that

$$\sum_{k=1}^{m_1} |a_k|^2 < \frac{\delta^2}{2} \quad \text{and} \quad \sup_{z \in L_1} |q_1(z) - g_1(z)| \leq 1.$$

We assume $a_{m_1} \neq 0$, and add all missing items $z^k, 1 \leq k \leq m_1$, with zero coefficients in $q_1(z)$.

Set $\mu_0 = 0$. Let

$$\mu_1 = \inf\{n \in \mu : n \geq \mu_0 + m_1\} < \infty.$$

Set $a_k = 0$ for $m_1 < k \leq \mu_1$, and define

$$p_{\mu_1}(z) = \sum_{k=1}^{m_1} a_k z^k + \sum_{k=m_1+1}^{\mu_1} a_k z^k.$$

Then and

$$\sum_{k=1}^{\mu_1} |a_k|^2 = \sum_{k=1}^{m_1} |a_k|^2 + 0 < \frac{\delta^2}{2},$$

$$\sup_{z \in L_1} |p_{\mu_1}(z) - g_1(z)| = \sup_{z \in L_1} |q_1(z) - g_1(z)| \leq 1.$$

Step 2. $n = 2$. Apply Lemma 3.1, there exists a polynomial $q_2(z) = \sum_{k=\mu_1+1}^{m_2} a_k z^k$ such that

$$\sum_{k=\mu_1+1}^{m_2} |a_k|^2 < \frac{\delta^2}{4} \quad \text{and} \quad \sup_{z \in L_2} |q_2(z) - (g_2(z) - p_{\mu_1}(z))| \leq 1/2.$$

We assume $a_{m_2} \neq 0$, and add all missing items z^k , $\mu_1 + 1 \leq k \leq m_2$, with zero coefficients in $q_2(z)$.

Let $\mu_2 = \inf\{n \in \mu : n \geq \mu_1 + m_2\} < \infty$. Set $a_k = 0$ for $m_2 < k \leq \mu_2$, and define

$$p_{\mu_2}(z) = \sum_{k=\mu_1+1}^{m_2} a_k z^k + \sum_{j=m_2+1}^{\mu_2} a_j z^j.$$

Then

$$\sum_{k=\mu_1+1}^{\mu_2} |a_k|^2 = \sum_{k=\mu_1+1}^{m_2} |a_k|^2 + 0 < \frac{\delta^2}{4},$$

$$\sup_{z \in L_2} |p_{\mu_2}(z) - (g_2(z) - p_{\mu_1}(z))| = \sup_{z \in L_2} |p_{\mu_1}(z) + p_{\mu_2}(z) - g_2(z)| < 1/2.$$

Step 3. Repetition. Repeat above processes, we obtain an increasing sequences $\{\mu_n\}_{n=1}^{\infty}$ in μ and a polynomial sequence of $\{p_{\mu_1}(z), p_{\mu_2}(z), \dots\}$, where $p_{\mu_n}(z) = \sum_{k=\mu_{n-1}+1}^{\mu_n} a_k z^k$ satisfying

$$\sum_{k=\mu_{n-1}+1}^{\mu_n} |a_k|^2 < \frac{\delta}{2^n}, \tag{3.3a}$$

$$\sup_{z \in L_n} \left| \sum_{i=1}^n p_{\mu_i}(z) - g_n(z) \right| < 1/n. \tag{3.3b}$$

Define

$$f(z) = \sum_{i=1}^{\infty} p_{\mu_i}(z) = \sum_{k=1}^{\infty} a_k z^k,$$

then $f(z)$ satisfies the requirement (i)–(iii).

First,

$$\sum_{k=1}^{\infty} |a_k|^2 = \sum_{n=1}^{\infty} \sum_{k=\mu_{n-1}+1}^{\mu_n} |a_k|^2 < \sum_{n=1}^{\infty} \frac{\delta}{2^n} = \delta,$$

so (iii) is satisfied.

Secondly, let $K \subset \{z \in \mathbb{C} : |z| > 1\}$ be a compact set having connected complement and h be a function continuous on K and holomorphic in the interior of K . Select a compact set K_m in the sequence $\{K_m\}_{m=1}^\infty$ such that $K \subset K_m$.

Let $\varepsilon_n \searrow 0$. Choose a polynomial f_j having coefficients with rational coordinates such that

$$\sup_{z \in K_m} |h(z) - f_j(z)| < \varepsilon_1/2. \tag{3.4}$$

By the construction of \mathcal{L} , the pair $(f_j; K_m)$ appears infinitely often, say appeared at $(g_{n_1}; L_{n_1}), (g_{n_2}; L_{n_2}), \dots$. Let $j_1 = \inf\{j : 1/n_j < \varepsilon_1/2\}$. By (3.3b)

$$\sup_{z \in L_{n_{j_1}}} \left| \sum_{i=1}^{n_{j_1}} p_{\mu_i}(z) - g_{n_{j_1}}(z) \right| \leq 1/n_{j_1} < \varepsilon_1/2. \tag{3.5}$$

Note that the count of total items in $\sum_{i=1}^{n_{j_1}} p_{\mu_i}(z)$ is

$$\sum_{i=1}^{n_{j_1}} \sum_{k=\mu_{i-1}+1}^{\mu_i} 1 = \sum_{i=1}^{n_{j_1}} (\mu_i - \mu_{i-1}) = \mu_{n_{j_1}}.$$

Set $\lambda_1 = \mu_{n_{j_1}}$. Then the λ_1 -th partial sum of $f(z)$ is

$$S_{\lambda_1}(f)(z) = S_{\mu_{n_{j_1}}}(f)(z) = \sum_{k=1}^{n_{j_1}} p_{\mu_k}(z).$$

Combining (3.4) and (3.5), we have

$$\begin{aligned} \sup_{z \in K} |S_{\lambda_1}(f)(z) - h(z)| &\leq \sup_{z \in K} |S_{\lambda_1}(f)(z) - f_j(z)| + \sup_{z \in K} |f_j(z) - h(z)| \\ &\leq \sup_{z \in L_{n_{j_1}}} |S_{\mu_{n_{j_1}}}(f)(z) - g_{n_{j_1}}(z)| + \sup_{z \in L_{n_{j_1}}} |f_j(z) - h(z)| \\ &< \varepsilon_1/2 + \varepsilon_1/2 = \varepsilon_1. \end{aligned}$$

Repeating the same procedure with the sequence $(g_{n_{j_1+1}}; L_{n_{j_1+1}}), (g_{n_{j_1+2}}; L_{n_{j_1+2}}), \dots$ and $\varepsilon_2 > 0$, we can get $\lambda_2 \in \mu, \lambda_2 > \lambda_1$ such that

$$\sup_{z \in K} |S_{\lambda_2}(f)(z) - h(z)| < \varepsilon_2.$$

Keep going we obtain a sequence $\{\lambda_n\}_{n=1}^\infty \subset \mu$ and

$$\lim_{n \rightarrow \infty} \sup_{z \in K} |S_{\lambda_n}(f)(z) - h(z)| = 0.$$

That is, the requirement (ii) is satisfied.

Thirdly, we show (i). The constrain inequality $\sum_{k=1}^{\infty} |a_k|^2 < \delta < 1$ guarantees $|a_k| < 1$ for $k = 1, 2, \dots$. Thus for every $|z| < 1$,

$$\left| \sum_{k=1}^{\infty} a_k z^k \right| \leq \sum_{k=1}^{\infty} |a_k| |z|^k < \sum_{k=1}^{\infty} |z|^k = \frac{|z|}{1 - |z|} < \infty.$$

For any $z_0 \in \mathbb{C}$ with $|z_0| > 1$. The singleton set $K_{z_0} = \{z_0\} \subset \{z \in \mathbb{C} : |z| > 1\}$ is compact with connected complement. Define function $f_1(z_0) = 1$ and $f_2(z_0) = -1$. Obviously, f_1 and f_2 are continuous on K_{z_0} and holomorphic in the interior of K_{z_0} . Apply the conclusion (ii) at f_1 on K_{z_0} and f_2 on K_{z_0} , respectively, there exist two increasing sequences $\{\lambda_{n_1}\}_{n_1=1}^{\infty}$ and $\{\lambda_{n_2}\}_{n_2=1}^{\infty}$ in μ such that

$$\lim_{n_1 \rightarrow \infty} S_{\lambda_{n_1}}(f)(z_0) = f_1(z_0) = 1 \quad \text{and} \quad \lim_{n_2 \rightarrow \infty} S_{\lambda_{n_2}}(f)(z_0) = f_2(z_0) = -1.$$

Thus the series $f(z) = \sum_{k=1}^{\infty} a_k z^k$ is divergent at z_0 whenever $|z_0| > 1$. Hence the radius of convergence of $\sum_{k=1}^{\infty} a_k z^k$ is exactly equal to 1. □

With the notations of $\{K_m\}_{m \geq 1}$ and $\{f_j\}_{j \geq 1}$ as above, for $0 < \delta < 1$ and any integers m, j, s, n with $m \geq 1, j \geq 1, s \geq 1$ and $n \geq 0$, we denote by $E_{\delta}(m, j, s, n)$ the set

$$E_{\delta}(m, j, s, n) = \left\{ g \in H_{\delta}(D) : \sup_{z \in K_m} |S_n(g)(z) - f_j(z)| < \frac{1}{s} \right\}.$$

Lemma 3.2. *Let $0 < \delta < 1$ and $\mu \subset \mathbb{N}$ be the infinite subset given in Theorem 2.1, then U_{δ}^{μ} can be written as follows:*

$$U_{\delta}^{\mu} = \bigcap_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n \in \mu} E_{\delta}(m, j, s, n).$$

Proof. The inclusion

$$U_{\delta}^{\mu} \subset \bigcap_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n \in \mu} E_{\delta}(m, j, s, n)$$

follows clearly from the definition of U_{δ}^{μ} and $E_{\delta}(m, j, s, n)$. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \bigcap_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n \in \mu} E_{\delta}(m, j, s, n),$$

we will show that $f \in U_{\delta}^{\mu}$. The definition of $E_{\delta}(m, j, s, n)$ guarantees $\sum_{n=0}^{\infty} |a_n|^2 < \delta$, so (iii) in Theorem 2.1 is satisfied automatically.

Let $K \subset \{z \in \mathbb{C} : |z| > 1\}$ be a non-empty compact set having connected complement and $h : K \rightarrow \mathbb{C}$ be a function continuous on K and holomorphic in the interior of K . To

prove (ii), it is sufficient to prove that for any $\varepsilon > 0$ and integer $N_0 > 1$ there exists integer $N > N_0$ such that

$$\sup_{z \in K} |S_N(f)(z) - h(z)| < \varepsilon. \tag{3.6}$$

The remaining of the proof is similar to the proof of [7, Lemma 2.2] except the argument on μ . Select a compact subset K_m in the sequence $\{K_m\}_{m \geq 1}$ such that $K \subset K_m$, choose a positive integer s such that $1/s < \varepsilon/2$, and a function f_j in the sequence $\{f_j\}_{j \geq 1}$ such that

$$\sup_{z \in K_m} |h(z) - f_j(z)| < \frac{1}{s} < \varepsilon/2. \tag{3.7}$$

As $f \in \bigcup_{n \in \mu} E_\delta(m, j, s, n)$, we choose the smallest $n_s \in \mu$, $n_s \geq 1$, such that

$$\sup_{z \in K_m} |S_{n_s}(f)(z) - f_j(z)| < \frac{1}{s} < \varepsilon/2.$$

If $n_s > N_0$, set $N = n_s$, and the job is done. If $n_s \leq N_0$, let

$$\varepsilon_1 = \frac{1}{s} - \sup_{z \in K_m} |S_{n_s}(f)(z) - f_j(z)| > 0.$$

Take an integer $s' > s$ such that $\frac{1}{s'} < \varepsilon_1$. As $f \in \bigcup_{n \in \mu} E_\delta(m, j, s', n)$, we can choose the smallest $n_{s'} \in \mu$, such that

$$\sup_{z \in K_m} |S_{n_{s'}}(f)(z) - f_j(z)| < \frac{1}{s'} < \varepsilon_1.$$

The selection ensure that $n_{s'} > n_s$. If $n_{s'} \leq N_0$ again, repeat the procedure at most N_0 times, finally we obtain a $n_{s^*} \geq N_0$, $n_{s^*} \in \mu$ such that

$$\sup_{z \in K_m} |S_{n_{s^*}}(f)(z) - f_j(z)| < \frac{1}{s} < \varepsilon/2, \tag{3.8}$$

and we let $N = n_{s^*} \in \mu$.

Combine (3.7), (3.8) and the fact that $K \subset K_m$, for the $N \in \mu$ and $N \geq N_0$, we have

$$\begin{aligned} \sup_{z \in K} |S_N(f)(z) - h(z)| &\leq \sup_{z \in K} |S_N(f)(z) - f_j(z)| + \sup_{z \in K} |h(z) - f_j(z)| \\ &\leq \sup_{z \in K_m} |S_N(f)(z) - f_j(z)| + \sup_{z \in K_m} |h(z) - f_j(z)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

That is (3.6), and (ii) in Theorem 2.1 is satisfied. By similar technique as in the proof of Lemma 2.2 in [7], we can prove that the convergent radius of the Taylor development of f is one, and (i) in Theorem 2.1 is verified. The proof of Lemma 3.2 is completed. \square

Note that our G_δ density is researched under the topology induced from Hardy space, and Mergelyan’s Theorem deal with the supremum norm only. Lemma 3.3 will build a bridge from supremum norm topology to Hardy space topology.

Lemma 3.3. *Let $f \in H^2(\mathbb{D})$. Let $0 < r < 1$. For every $\varepsilon > 0$,*

$$\{g \in H^2 : \|f - g\|_{H^2} < \varepsilon(1 - r)\} \subseteq \left\{g \in H^2 : \sup_{|z| \leq r} |f(z) - g(z)| < \varepsilon\right\}.$$

Proof. Suppose that g satisfies $\|f - g\|_{H^2} < \varepsilon(1 - r)$. By Cauchy integral formula and Cauchy inequality,

$$\begin{aligned} |f(re^{it}) - g(re^{it})| &= \left| \frac{1}{2\pi i} \int_{\tilde{\xi} \in \partial \mathbb{D}} \frac{f(\tilde{\xi}) - g(\tilde{\xi})}{\tilde{\xi} - re^{it}} d\tilde{\xi} \right| \\ &= \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(e^{is}) - g(e^{is})}{e^{is} - re^{it}} ie^{is} ds \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{is}) - g(e^{is})|}{1 - r} ds \\ &\leq \frac{1}{1 - r} \|f - g\|_{H^2} < \frac{1}{1 - r} \varepsilon(1 - r) = \varepsilon. \end{aligned}$$

Apply the Maximum modulus principle,

$$\sup_{|z| \leq r} |f(z) - g(z)| < \varepsilon.$$

Thus g is on the right side. □

Lemma 3.4. *Let $0 < \delta < 1$, then for every integer $m \geq 1, j \geq 1, s \geq 1$, and $n \geq 0$, the set $E_\delta(m, j, s, n)$ is open in $H_\delta(\mathbb{D})$.*

Proof. Let $f(z) = \sum_{k=0}^\infty a_k z^k \in E_\delta(m, j, s, n)$. Then

$$\sup_{z \in K_m} |S_n(f)(z) - f_j(z)| < \frac{1}{s} \quad \text{and} \quad \|f\|_{H^2} = \left(\sum_{k=0}^\infty |a_k|^2 \right)^{1/2} < \delta.$$

It’s sufficient to find a number $0 < \alpha < 1/2$, such that the condition $g(z) = \sum_{k=0}^\infty b_k z^k \in H^2(\mathbb{D})$ and $\|f - g\|_{H^2} < \alpha$ will imply $g \in E_\delta(m, j, s, n)$, i.e.,

$$\|g\|_{H^2} = \left(\sum_{k=0}^\infty |b_k|^2 \right)^{1/2} < \delta$$

and

$$\sup_{z \in K_m} |S_n(g)(z) - f_j(z)| < \frac{1}{s}. \tag{3.9}$$

Let $\alpha_1 = \delta - (\sum_{k=0}^{\infty} |a_k|^2)^{1/2} > 0$. Then $\|g - f\|_{H^2} < \alpha_1$ implies

$$\|g\|_{H^2} < \|f\|_{H^2} + \alpha_1 = \delta.$$

Let $M = \sup_{z \in K_m} |z|$, then $1 < M < \infty$. Set

$$\alpha_2 = \left(\frac{1}{s} - \sup_{z \in K_m} |S_n(f)(z) - f_j(z)| \right) \left(\sum_{l=1}^n 2^l M^l \right)^{-1} > 0.$$

Applying the same reasoning argument as in the proof of Lemma 2.3 in [7], we can prove

$$\begin{aligned} & \left\{ g \in H(\mathbb{D}) : \sup_{|z| \leq \frac{1}{2}} |g(z) - f(z)| < \alpha_2 \right\} \\ & \subseteq \left\{ g \in H(\mathbb{D}) : \sup_{z \in K_m} |S_n(g)(z) - f_j(z)| < \frac{1}{s} \right\}. \end{aligned} \tag{3.10}$$

Take $r = 1/2$ and $\varepsilon = \beta > 0$ in Lemma 3.3, we have

$$\{g \in H^2 : \|f - g\|_{H^2} < \alpha_2/2\} \subseteq \left\{ g \in H^2 : \sup_{|z| \leq 1/2} |f(z) - g(z)| < \alpha_2 \right\}. \tag{3.11}$$

Let $\alpha = \min\{\alpha_1, \alpha_2/2\}$. Combining (3.10) and (3.11) together, we have

$$\begin{aligned} \{g \in H^2 : \|f - g\|_{H^2} < \alpha\} & \subseteq \left\{ g \in H_\delta(\mathbb{D}) : \sup_{z \in K_m} |S_n(g)(z) - f_j(z)| < \frac{1}{s} \right\} \\ & = E_\delta(m, j, s, n). \end{aligned}$$

That is, (3.9) holds true. The proof of Lemma 3.4 is completed. □

Lemma 3.5. *Let $0 < \delta < 1$ and $\mu \subset \mathbb{N}$ be the infinite set. Then for every integer $m \geq 1, j \geq 1$, and $s \geq 1$, the set $\bigcup_{n \in \mu} E_\delta(m, j, s, n)$ is open and dense in $H_\delta(\mathbb{D})$.*

Proof. By Lemma 3.4, the sets $E_\delta(m, j, s, n), n = 0, 1, 2, \dots$ are open in $H_\delta(\mathbb{D})$. Therefore the same is true for the union $\bigcup_{n \in \mu} E_\delta(m, j, s, n)$. We will prove that this set is dense, i.e., for every $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H_\delta(\mathbb{D})$ and every $\varepsilon > 0$, there exists $g \in \bigcup_{n \in \mu} E_{2,\delta}(m, j, s, n)$ such that $\|f - g\|_{H^2} < \varepsilon$.

Since $f \in H_\delta(\mathbb{D})$, we can choose integer $m > 0$ such that

$$\left(\sum_{n=m+1}^{\infty} |a_n|^2 \right)^{1/2} < \varepsilon/2.$$

Apply Lemma 3.1 to the function

$$f_j(z) - \sum_{n=0}^m a_n z^n, \quad \varepsilon = 1/s \quad \text{and} \quad \delta_1 = \min \left\{ \varepsilon/2, \left(\delta - \sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2} \right\},$$

there exists a polynomial $h(z) = \sum_{k=0}^r c_k z^k$ such that

$$\left(\sum_{k=0}^r |c_k|^2 \right)^{1/2} < \delta_1,$$

$$\sup_{z \in K_m} \left| h(z) - \left(f_j(z) - \sum_{n=0}^m a_n z^n \right) \right| = \sup_{z \in K_m} \left| \sum_{k=0}^r c_k z^k - \left(f_j(z) - \sum_{k=0}^m a_k z^k \right) \right| < 1/s.$$

Without loss of generality, assume $r > m$. Choose $n \in \mu$ large enough such that $n > r$. Define $\{b_j\}_{0 \leq j \leq \mu_n}$ by

$$b_j = \begin{cases} a_j + c_j, & 0 \leq j \leq m, \\ c_j, & m < j \leq r, \\ 0, & r < j \leq n, \end{cases}$$

and define $g(z) = \sum_{k=0}^n b_k z^k$. Then

$$\sup_{z \in K_m} |g(z) - f_j(z)| = \sup_{z \in K_m} \left| \sum_{k=0}^r c_k z^k - \left(f_j(z) - \sum_{k=0}^m a_k z^k \right) \right| < 1/s.$$

And

$$\begin{aligned} \left(\sum_{k=0}^{\mu_n} |b_k|^2 \right)^{1/2} &\leq \left(\sum_{k=0}^m |a_k|^2 \right)^{1/2} + \left(\sum_{k=0}^r |c_k|^2 \right)^{1/2} \\ &\leq \left(\sum_{k=0}^{\infty} |a_k|^2 \right)^{1/2} + \left(\sum_{k=0}^r |c_k|^2 \right)^{1/2} \\ &< \left(\sum_{k=0}^{\infty} |a_k|^2 \right)^{1/2} + \delta - \left(\sum_{k=0}^{\infty} |a_k|^2 \right)^{1/2} = \delta. \end{aligned}$$

Thus $g \in E_\delta(m, j, s, n)$. We also have

$$\begin{aligned} \|g(z) - f(z)\|_{H^2} &\leq \left\| g(z) - \sum_{n=0}^m a_n z^n \right\|_{H^2} + \left\| \sum_{n=0}^m a_n z^n - f(z) \right\|_{H^2} \\ &\leq \left(\sum_{k=0}^r |c_k|^2 \right)^{1/2} + \left(\sum_{n=m+1}^{\infty} |a_n|^2 \right)^{1/2} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This completes the proof. □

Now we prove Theorem 2.2.

Proof of Theorem 2.2. For every $0 < \delta < 1$ and infinite subset μ of \mathbb{N} , by Lemma 3.2,

$$U_{\delta}^{\mu} = \bigcap_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n \in \mu} E_{\delta}(m, j, s, n).$$

By Lemmas 3.4 and 3.5, $\bigcup_{n \in \mu} E_{\delta}(m, j, s, n)$ is open and dense in $H_{\delta}(D)$. As an open subset of the complete metric space $H^2(\mathbb{D})$, $H_{\delta}(\mathbb{D})$ is a Baire space. Hence $U_{2, \delta}^{\mu}$ is a non-empty G_{δ} dense subset in $H_{\delta}(\mathbb{D})$. The proof of Theorem 2.2 is completed. \square

4 Conclusions

Bayart, Grosse-Erdmann, Nestoridis and Papadimitropoulos [1] built abstract models for restricted universal series. From these models, they deduced many existing results as well as new statements in a unified way.

Although our δ -universal series could have overlap with abstract models in [1], it deserve a special attention and further research as the following reasons.

First, it seems that no one in the research of restricted universal series has considered the *overall constraints* of the coefficients. Think about what we proved. A series $\sum_{k=1}^{\infty} a_k z^k$ with very small $\sum_{k=1}^{\infty} |a_k|^2$ can approximate functions with huge modular. What a surprise! Secondly, the G_{δ} density does not hold in the entire space, but is confined to an open ball with an arbitrarily small radius, which is completely different from the result in [1]. Thirdly, the existence of universal series is essentially the existence of a good form (starting index greater than arbitrary integer) of approximation theorem [4, Proposition 7, Page 353]; and the existence of δ -universal series is essentially the existence of a strong form (starting index greater than arbitrary integer and $\sum_{k=1}^{\infty} |a_k|^2 < \delta$) of approximation theorem. Finding a strong form for existed approximation theorems has interest in its own right.

Acknowledgements

This research was supported by the National Natural Science Foundation of China (Grant No. 12401127).

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