Analysis in Theory and Applications
Anal. Theory Appl., Vol. **42**, No. 1 (2026), pp. 62-89

DOI: 10.4208/ata.2026.deng90.02
March 2026

Signal and Image Recovery with Scale and Signed Permutation Invariant Sparsity-Promoting Functions

Jianqing Jia[1], Ashley Prater-Bennette[2] and Lixin Shen[1,*]

[1] *Department of Mathematics, Syracuse University, Syracuse, NY 13244, USA*
[2] *Air Force Research Laboratory, Rome, NY 13441, USA*

Received 28 December 2024; Accepted (in revised version) 19 July 2025

Dedicated to the memory of Prof. Donggao Deng on the occasion of his 90th birthday

---

Abstract. Sparse signal recovery has been a cornerstone of advancements in data processing and imaging. Recently, the squared ratio of ℓ_1 to ℓ_2 norms, $(\ell_1/\ell_2)^2$, has been introduced as a sparsity-prompting function, showing superior performance compared to traditional ℓ_1 minimization, particularly in challenging scenarios with high coherence and dynamic range. This paper explores the integration of the proximity operator of $(\ell_1/\ell_2)^2$ and ℓ_1/ℓ_2 into efficient optimization frameworks, including the Accelerated Proximal Gradient (APG) and Alternating Direction Method of Multipliers (ADMM). We rigorously analyze the convergence properties of these algorithms and demonstrate their effectiveness in compressed sensing and image restoration applications. Numerical experiments highlight the advantages of our proposed methods in terms of recovery accuracy and computational efficiency, particularly under noise and high-coherence conditions.

Key Words: Sparse signal recovery, compressed sensing, image restoration, $(\ell_1/\ell_2)^2$, optimization algorithms

AMS Subject Classifications: 90C26, 65K05

---

1 Introduction

In the era of big data, sparse learning has emerged as a transformative paradigm, enabling efficient representation and processing of inherently sparse datasets. Sparse modeling techniques have proven particularly valuable in fields such as signal processing, compressed sensing, and image restoration, where the goal is often to recover high-dimensional signals with minimal active components.

*Corresponding author. *Email addresses:* jjia10@syr.edu (J. Jia), ashley.prater-bennette@us.af.mil (A. Prater-Bennette), lshen03@syr.edu (L. Shen)

Among the array of sparsity-promoting functions, the ratio of ℓ_1 to ℓ_2 norms, specifically its squared form $(\ell_1/\ell_2)^2$, has gained attention for its robustness in handling high-coherence and dynamic range scenarios. Unlike traditional minimization ℓ_1, this approach dynamically balances sparsity and energy, resulting in superior recovery performance in complex environments. Despite its potential, integrating this function into an optimization framework has been non-trivial, primarily due to its nonconvexity and computational challenges.

This paper builds on the foundation work in [5,6], where the proximity operator of $(\ell_1/\ell_2)^2$ and ℓ_1/ℓ_2 were efficiently computed. Leveraging this breakthrough, we integrate the proximity operator into two efficient optimization methods: the Accelerated Proximal Gradient (APG) and the Alternating Direction Method of Multipliers (ADMM). Our contributions are threefold:

1. We present novel formulations for incorporating $(\ell_1/\ell_2)^2$ into APG and ADMM frameworks.

2. We rigorously analyze the convergence properties of these methods under various setting.

3. We conduct extensive numerical experiments demonstrating the superior performance of our proposed algorithms in sparse signal recovery and image restoration tasks.

The remainder of this paper is organized as follows: Section 2 provides the necessary mathematical preliminaries and background. Section 3 elaborates on the application of the proposed methods to sparse signal recovery, including algorithm details and theoretical analysis. Section 4 extends the discussion to image restoration, showcasing practical implementations and results. Finally, Section 5 concludes the paper by summarizing key findings and highlighting future research directions.

2 Preliminaries

All functions in this work are defined on Euclidean space $\mathbb{R}^n$. Bold lowercase letters, such as $\boldsymbol{x}$, signify vectors, with the jth component represented by the corresponding lowercase letter x_j. The notation $\mathrm{supp}(\boldsymbol{x})$ denotes the support of the vector $\boldsymbol{x}$, defined as $\mathrm{supp}(\boldsymbol{x}) = \{k : x_k \neq 0\}$. Matrices are indicated by bold uppercase letters such as A and B. Let $\mathcal{P}_n$ denote the set of all $n \times n$ signed permutation matrices: those matrices that have only one nonzero entry in every row or column, which is ± 1.

The ℓ_p norm of $\boldsymbol{x} = [x_1, \cdots, x_n]^\top \in \mathbb{R}^n$ is defined as $\|\boldsymbol{x}\|_p = (\sum_{k=1}^n |x_k|^p)^{1/p}$ for $1 \leq p < \infty$, $\|\boldsymbol{x}\|_\infty = \max_{1\leq k\leq n} |x_k|$, and $\|\boldsymbol{x}\|_0$ being the number of non-zero components in $\boldsymbol{x}$. The Frobenius norm of matrix A is defined as $\|\mathsf{A}\|_F = (\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2)$. The standard inner product in $\mathbb{R}^n$ is denoted by $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^n$.