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$L^p \rightarrow L^q$ Estimates for Stein's Spherical Maximal Operators

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Dedicated to the memory of Prof. Donggao Deng on the occasion of his 90th birthday

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Abstract. In this article we consider a modification of the Stein's spherical maximal operator of complex order α on $\mathbb{R}^n$:

$$\mathfrak{M}^{\alpha}_{[1,2]} f(x) = \sup_{t\in[1,2]} \left| \frac{1}{\Gamma(\alpha)} \int_{|y|\le 1} \left(1-|y|^2\right)^{\alpha-1} f(x-ty)dy \right|.$$

We show that when $n \ge 2$, suppose $\left\|\mathfrak{M}^{\alpha}_{[1,2]} f\right\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}$ holds for some $\alpha \in \mathbb{C}$, $p, q \ge 1$, then we must have that $q \ge p$ and

$$\operatorname{Re}\alpha \ge \sigma_n(p,q) := \max\left\{ \frac{1}{p} - \frac{n}{q}, \frac{n+1}{2p} - \frac{n-1}{2}\left(\frac{1}{q}+1\right), \frac{n}{p} - n + 1 \right\}.$$

Conversely, we show that $\mathfrak{M}^{\alpha}_{[1,2]}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ provided that $q \ge p$ and $\operatorname{Re}\alpha > \sigma_2(p,q)$ for $n=2$; and

$$\operatorname{Re}\alpha > \max\left\{\sigma_n(p,q),\ 1/(2p) - (n-2)/(2q) - (n-1)/4\right\}$$

for $n > 2$. The range of α, p and q is almost optimal in the case when either $n=2$, or $\alpha = 0$, or (p,q) lies in certain regions for $n > 2$.

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1 Introduction

In 1976, Stein [22] introduced the spherical maximal means $\mathfrak{M}^\alpha f(x) = \sup_{t>0} |\mathfrak{M}_t^\alpha f(x)|$ of (complex) order α on $\mathbb{R}^n$, where

$$\mathfrak{M}_t^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{|y|\le 1} \left(1-|y|^2\right)^{\alpha-1} f(x-ty)\,\mathrm{d}y. \tag{1.1}$$

These means are initially defined only for Re $\alpha > 0$, but the definition can be extended to all complex α by analytic continuation. In the case $\alpha = 1$, $\mathfrak{M}^\alpha$ corresponds to the Hardy-Littlewood maximal operator and in the case $\alpha = 0$, $\mathfrak{M}^\alpha$ corresponds to the spherical maximal operator $\mathfrak{M}f(x) = \sup_{t>0} |\mathfrak{M}_t f(x)|$ in which

$$\mathfrak{M}_t f(x) = c_n \int_{\mathbb{S}^{n-1}} f(x-ty)\,\mathrm{d}\sigma(y), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $\mathbb{S}^{n-1}$ denotes the standard unit sphere in $\mathbb{R}^n$. In [22] Stein obtained the inequality

$$\|\mathfrak{M}^\alpha f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \tag{1.3}$$

for $\mathrm{Re}\,\alpha > 1-n+n/p$ when $1 < p \le 2$; or $\mathrm{Re}\,\alpha > (2-n)/p$ when $2 \le p \le \infty$. From it, we see that when $\alpha = 0$ and $n \ge 3$, the maximal operator $\mathfrak{M}$ is bounded on $L^p(\mathbb{R}^n)$ for the range $p > n/(n-1)$. This range of p is sharp, as has been pointed out in [22, 24], no such result can hold for $p \le n/(n-1)$ if $n \ge 2$. The extension of this result in [22] to the case $n = 2$ was established about a decade later by Bourgain [2], see also the account in [23, Chapter XI].

In addition to Stein and Bourgain, other authors have studied the spherical maximal means; for instance see [10–13, 15–18, 20, 21, 28] and the references therein. All these refinements can be stated altogether as follows: When $n \ge 2$, suppose (1.3) holds for some α and $p \ge 2$, then we must have that $\mathrm{Re}\,\alpha \ge \max\{1/p-(n-1)/2,\ -(n-1)/p\}$. Further, the estimate (1.3) holds whenever $p \ge 2$ and

$$\mathrm{Re}\,\alpha > \begin{cases} \max\left\{\dfrac{1}{p}-\dfrac{1}{2}, \dfrac{1-n}{p}\right\}, & n=2, \\ \max\left\{\dfrac{1-n}{4}+\dfrac{3-n}{2p}, \dfrac{1-n}{p}\right\}, & n \ge 3. \end{cases} \tag{1.4}$$

1.1 Main results

In this article we modify the definition of the Stein's spherical maximal operator $\mathfrak{M}^\alpha$ so that the supremum is taken over, say, $1 \le t \le 2$, i.e.,

$$\mathfrak{M}^\alpha_{[1,2]} f(x) := \sup_{t\in[1,2]} \left|\mathfrak{M}_t^\alpha f(x)\right|,$$

then the resulting maximal function is actually bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some $q > p$. More precisely, we have the following results.