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$L^p \to L^q$ Estimates for Stein's Spherical Maximal Operators

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Dedicated to the memory of Prof. Donggao Deng on the occasion of his 90th birthday

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Abstract. In this article we consider a modification of the Stein's spherical maximal operator of complex order α on $\mathbb{R}^n$:

$$\mathfrak{M}^\alpha_{[1,2]} f(x) = \sup_{t\in[1,2]} \left| \frac{1}{\Gamma(\alpha)} \int_{|y|\le 1} \left(1-|y|^2\right)^{\alpha-1} f(x-ty)dy \right|.$$

We show that when $n \geq 2$, suppose $\|\mathfrak{M}^\alpha_{[1,2]} f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$ holds for some $\alpha \in \mathbb{C}$, $p, q \geq 1$, then we must have that $q \geq p$ and

$$\mathrm{Re}\,\alpha \geq \sigma_n(p,q) := \max\left\{ \frac{1}{p} - \frac{n}{q}, \frac{n+1}{2p} - \frac{n-1}{2}\left(\frac{1}{q}+1\right), \frac{n}{p} - n + 1 \right\}.$$

Conversely, we show that $\mathfrak{M}^\alpha_{[1,2]}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ provided that $q \geq p$ and $\mathrm{Re}\,\alpha > \sigma_2(p,q)$ for $n = 2$; and

$$\mathrm{Re}\,\alpha > \max\left\{\sigma_n(p,q),\ 1/(2p) - (n-2)/(2q) - (n-1)/4\right\}$$

for $n > 2$. The range of α, p and q is almost optimal in the case when either $n = 2$, or $\alpha = 0$, or (p,q) lies in certain regions for $n > 2$.

Key Words: spherical maximal operators, L^p-improving estimates, wave equation, local smoothing estimates

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1 Introduction

In 1976, Stein [22] introduced the spherical maximal means $\mathfrak{M}^\alpha f(x) = \sup_{t>0} |\mathfrak{M}_t^\alpha f(x)|$ of (complex) order α on $\mathbb{R}^n$, where

$$\mathfrak{M}_t^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{|y|\le 1} \left(1-|y|^2\right)^{\alpha-1} f(x-ty)\,\mathrm{d}y. \tag{1.1}$$

These means are initially defined only for Re $\alpha > 0$, but the definition can be extended to all complex α by analytic continuation. In the case $\alpha = 1$, $\mathfrak{M}^\alpha$ corresponds to the Hardy-Littlewood maximal operator and in the case $\alpha = 0$, $\mathfrak{M}^\alpha$ corresponds to the spherical maximal operator $\mathfrak{M}f(x) = \sup_{t>0} |\mathfrak{M}_t f(x)|$ in which

$$\mathfrak{M}_t f(x) = c_n \int_{\mathbb{S}^{n-1}} f(x-ty)\,\mathrm{d}\sigma(y), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $\mathbb{S}^{n-1}$ denotes the standard unit sphere in $\mathbb{R}^n$. In [22] Stein obtained the inequality

$$\|\mathfrak{M}^\alpha f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \tag{1.3}$$

for $\mathrm{Re}\,\alpha > 1-n+n/p$ when $1 < p \le 2$; or $\mathrm{Re}\,\alpha > (2-n)/p$ when $2 \le p \le \infty$. From it, we see that when $\alpha = 0$ and $n \ge 3$, the maximal operator $\mathfrak{M}$ is bounded on $L^p(\mathbb{R}^n)$ for the range $p > n/(n-1)$. This range of p is sharp, as has been pointed out in [22, 24], no such result can hold for $p \le n/(n-1)$ if $n \ge 2$. The extension of this result in [22] to the case $n = 2$ was established about a decade later by Bourgain [2], see also the account in [23, Chapter XI].

In addition to Stein and Bourgain, other authors have studied the spherical maximal means; for instance see [10–13, 15–18, 20, 21, 28] and the references therein. All these refinements can be stated altogether as follows: When $n \ge 2$, suppose (1.3) holds for some α and $p \ge 2$, then we must have that $\mathrm{Re}\,\alpha \ge \max\{1/p-(n-1)/2,\ -(n-1)/p\}$. Further, the estimate (1.3) holds whenever $p \ge 2$ and

$$\mathrm{Re}\,\alpha > \begin{cases} \max\left\{\dfrac{1}{p}-\dfrac{1}{2}, \dfrac{1-n}{p}\right\}, & n=2, \\ \max\left\{\dfrac{1-n}{4}+\dfrac{3-n}{2p}, \dfrac{1-n}{p}\right\}, & n\ge 3. \end{cases} \tag{1.4}$$

1.1 Main results

In this article we modify the definition of the Stein's spherical maximal operator $\mathfrak{M}^\alpha$ so that the supremum is taken over, say, $1 \le t \le 2$, i.e.,

$$\mathfrak{M}^\alpha_{[1,2]} f(x) := \sup_{t\in[1,2]} \left|\mathfrak{M}_t^\alpha f(x)\right|,$$

then the resulting maximal function is actually bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some $q > p$. More precisely, we have the following results.

Theorem 1.1. *Let $n \geq 2$ and $p, q \in [1,\infty]$. Suppose*

$$\left\|\mathfrak{M}^{\alpha}_{[1,2]} f\right\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \tag{1.5}$$

holds for some $\alpha \in \mathbb{C}$. Then we must have $q \geq p$ and

$$\operatorname{Re}\alpha \geq \sigma_n(p,q) := \max\left\{\frac{1}{p}-\frac{n}{q},\ \frac{n+1}{2p}-\frac{n-1}{2}\left(\frac{1}{q}+1\right),\frac{n}{p}-n+1\right\}.$$

Theorem 1.2. *Assume $q \geq p$.*

(i) *Let $n = 2$. If $\operatorname{Re}\alpha > \sigma_2(p,q)$, then the estimate* (1.5) *holds.*

(ii) *Let $n > 2$. If*

$$\operatorname{Re}\alpha > d_n(p,q) := \max\left\{\sigma_n(p,q),\ \frac{1}{2p}-\frac{n-2}{2q}-\frac{n-1}{4}\right\},$$

then the estimate (1.5) *holds.*

From Theorem 1.1 and (i) of Theorem 1.2, we see that when $n = 2$, the range of α is sharp except for the boundary case $\operatorname{Re}\alpha = \sigma_2(p,q)$ when $q \geq p$.

Note that if $(1/p, 1/q)$ belongs to the set $\triangle ODE \backslash \triangle ABC$ (see Fig. 1 below), then $\sigma_n(p,q) \geq \frac{1}{2p} - \frac{n-2}{2q} - \frac{n-1}{4}$, and so $d_n(p,q) = \sigma_n(p,q)$.

It then follows from Theorem 1.1 and (ii) of Theorem 1.2 that if $n > 2$, the range of α is sharp when $(1/p, 1/q)$ belongs to the set $\triangle ODE \backslash \triangle ABC$ except for the boundary case $\operatorname{Re}\alpha = \sigma_n(p,q)$.

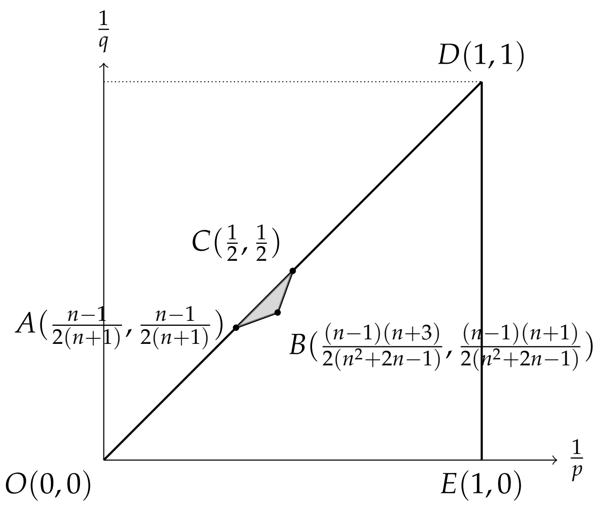

Figure 1: The range of $(1/p, 1/q)$ in (ii) of Theorem 1.2.

When $n > 2$ and $\alpha = 0$, (ii) of Theorem 1.2 implies that the result of Schlag and Sogge [18], which is optimal except the boundaries. Indeed, in the case $\alpha = 0$ almost sharp results about the spherical maximal operators $\mathfrak{M}_{[1,2]}$ have been obtained in Schlag [17] for $n=2$, and Schlag and Sogge [18] for $n \geq 2$. Recall that from [17,18], a necessary condition for $L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ boundedness of $\mathfrak{M}_{[1,2]}$ is that $(1/p, 1/q)$ belongs to the closed quadrangle $\mathcal{Q}$ with corners $P_1 = (0,0)$, $P_2 = (\frac{n-1}{n}, \frac{n-1}{n})$, $P_3 = (\frac{n-1}{n}, \frac{1}{n})$ and $P_4 = (\frac{n(n-1)}{n^2+1}, \frac{n-1}{n^2+1})$ (when $n=2$, the quadrangle $\mathcal{Q}$ becomes a triangle as the points P_2 and P_3 coincide); further, if $(1/p, 1/q)$ belongs to the interior of $\mathcal{Q}$, then $\mathfrak{M}_{[1,2]}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

1.2 About our method

Let us outline the proof of Theorems 1.1 and 1.2. To show Theorem 1.1, we employ the asymptotic expansion of the Fourier multiplier associated with the operator $\mathfrak{M}_t^\alpha$ to see that $\mathfrak{M}_t^\alpha$ are essentially the linear combination of half-wave operators $e^{it\sqrt{-\Delta}}\langle D\rangle^{-\frac{n-1}{2}-\alpha}$ and $e^{-it\sqrt{-\Delta}}\langle D\rangle^{-\frac{n-1}{2}-\alpha}$. Consequently, the complexity of the operator $\mathfrak{M}_t^\alpha$ arises from the interference between the operators $e^{it\sqrt{-\Delta}}$ and $e^{-it\sqrt{-\Delta}}$. To show the necessity of $L^p \to L^q$ boundedness of $\mathfrak{M}_t^\alpha$ in Theorem 1.1, we construct three special examples such that there is no interference between $e^{it\sqrt{-\Delta}}f$ and $e^{-it\sqrt{-\Delta}}f$.

The proof Theorem 1.2 is shown by combining $L^p \to L^q$ local smoothing estimates for wave operators, and the techniques previously used in [13] and [12]. To obtain $L^p \to L^q$ local smoothing estimates for wave operators, we will apply $L^p \to L^p$ local smoothing estimates for wave operators and interpolation. We mention that local smoothing conjecture was originally formulated by Sogge [21]: for $n \geq 2$ and $p \geq 2n/(n-1)$, one has

$$\|u\|_{L^p(\mathbb{R}^n\times[1,2])} \leq C\left(\|f\|_{W^{\gamma,p}(\mathbb{R}^n)} + \|g\|_{W^{\gamma-1,p}(\mathbb{R}^n)}\right), \quad \text{if } \gamma > \frac{n-1}{2} - \frac{n}{p},$$

where

$$u(x,t) = \cos(t\sqrt{-\Delta})f(x) + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}g(x).$$

The local smoothing conjecture has been studied in numerous papers, see for instance [3, 6, 8, 9, 12, 13, 20, 27] and the references therein. When $n=2$, sharp results were established by Guth, Wang and Zhang [8]. When $n \geq 3$, the conjecture holds for all $p \geq 2(n+1)/(n-1)$ by the Bourgain-Demeter decoupling theorem [3] and the method of [27]. Up to now, the conjecture is still open in the case $2n/(n-1) \leq p < 2(n+1)/(n-1)$ and $n > 2$.

The paper is organized as follows. In Section 2, we give some basic results including the properties of the Fourier multiplier associated to the spherical operators $\mathfrak{M}_t^\alpha$ by using asymptotic expansions of Bessel functions. The proof of Theorem 1.1 will be given in Section 3 by constructing three examples to obtain the necessarity of $L^p \to L^q$ boundedness for the maximal operator $\mathfrak{M}^{\alpha}_{[1,2]}$. The proof of Theorem 1.2 will be given in Section 4.

2 Preliminary results

Let $\Gamma(\alpha)$ be the Gamma function and $(r)_+ = \max\{r, 0\}$ for $r \in \mathbb{R}$. Recall that the spherical function is defined by $\mathfrak{M}^{\alpha}_t f(x) = f * m_{\alpha,t}(x)$, where $m_{\alpha,t}(x) = t^{-n} m_\alpha(t^{-1}x)$ and

$$m_\alpha(x) = \Gamma(\alpha)^{-1}\big(1-|x|^2\big)^{\alpha-1}_+.$$

Define the Fourier transform of f by $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} f(x)\,dx$. It follows by [25, p.171] that the Fourier transform of m_α is given by

$$\widehat{m_\alpha}(\xi) = \pi^{-\alpha+1}|\xi|^{-n/2-\alpha+1} J_{n/2+\alpha-1}\big(2\pi|\xi|\big). \tag{2.1}$$

Here J_β denotes the Bessel function of order β. For any complex number β, we can obtain the complete asymptotic expansion

$$J_\beta(r) \sim r^{-1/2} e^{ir} \sum_{j=0}^{\infty} b_j r^{-j} + r^{-1/2} e^{-ir} \sum_{j=0}^{\infty} d_j r^{-j}, \quad r \geq 1, \tag{2.2}$$

for suitable coefficients b_j and d_j with $b_0, d_0 \neq 0$. More precisely, there exists error terms $\{E_{N,1}\}_{N\geq 1}$ and $\{E_{N,2}\}_{N\geq 1}$ such that for any given $N \geq 1$ and $r \geq 1$, there holds

$$J_\beta(r) = r^{-1/2} e^{ir} \left(\sum_{j=0}^{N-1} b_j r^{-j} + E_{N,1}(r) \right) + r^{-1/2} e^{-ir} \left(\sum_{j=0}^{N-1} d_j r^{-j} + E_{N,2}(r) \right) \tag{2.3}$$

and

$$\left| \left(\frac{d}{dr} \right)^k E_{N,1}(r) \right| + \left| \left(\frac{d}{dr} \right)^k E_{N,2}(r) \right| \leq C_k r^{-N-k}, \tag{2.4}$$

for all $k \in \mathbb{Z}_+$. Note that when β is a positive integer, the above results are given in [23, (15), p.338]. For a general β, we refer to [26, Chapter 7.2].

We rewrite (2.1) as

$$\begin{aligned} \widehat{m_\alpha}(\xi) =& \varphi(|\xi|)\widehat{m_\alpha}(\xi) + (1-\varphi(|\xi|))\widehat{m_\alpha}(\xi) \\ =& \varphi(|\xi|)\widehat{m_\alpha}(\xi) + \left[e^{2\pi i|\xi|}\mathcal{E}_{N,1}(|\xi|) + e^{-2\pi i|\xi|}\mathcal{E}_{N,2}(|\xi|) \right] \\ &+ |\xi|^{-(n-1)/2-\alpha} \left[e^{2\pi i|\xi|} a_1(|\xi|) + e^{-2\pi i|\xi|} a_2(|\xi|) \right], \end{aligned} \tag{2.5}$$

where

$$\mathcal{E}_{N,\ell}(r) = c(\pi,\alpha)E_{N,\ell}(2\pi r)(1-\varphi(r))r^{-(n-1)/2-\alpha}, \quad \ell = 1,2, \tag{2.6a}$$

$$a_1(r) = c(\pi,\alpha)\sum_{j=0}^{N-1} b_j(2\pi r)^{-j}(1-\varphi(r)), \tag{2.6b}$$

$$a_2(r) = c(\pi,\alpha)\sum_{j=0}^{N-1} d_j(2\pi r)^{-j}(1-\varphi(r)), \tag{2.6c}$$

with $c(\pi,\alpha) = 2^{-1/2}\pi^{-\alpha+1/2}$. Here $\varphi \in C_0^\infty(\mathbb{R})$ is an even function, identically equals 1 on $B(0,M)$ and supported on $B(0,2M)$, where $M = M(N)$ is large enough such that

$$\inf_{i=1,2}\inf_{|r|\geq M}|a_i(r)| \geq c_{low} > 0 \tag{2.7}$$

and there exist $\{\theta_i\}_{i=1,2} \subseteq [0,2\pi)$ such that

$$\sup_{i=1,2}\sup_{|r|\geq M}|\arg a_i(r) - \theta_i| \leq 10^{-2}. \tag{2.8}$$

Then we can split the Fourier multiplier of the operator $\mathfrak{M}_1^\alpha$ into three parts as in (2.5) above.

Lemma 2.1. *For $q \geq p \geq 1$, we have*

$$\Big\| \sup_{t\in[1,2]} |\widehat{m_\alpha}(tD)\varphi(t|D|)f| \Big\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}. \tag{2.9}$$

Proof. For all $\tau \geq 1$ and $t \in [1,2]$, we note that $\varphi(t|\xi|)\widehat{m_\alpha}(t\xi)\langle\xi\rangle^\tau \in C_c^\infty(\mathbb{R}^n)$. Then by Young's inequality and Sobolev embedding, we have

$$\begin{aligned}\Big\| \sup_{t\in[1,2]} |\widehat{m_\alpha}(tD)\varphi(t|D|)f| \Big\|_{L^q(\mathbb{R}^n)} &\leq C\|\langle\cdot\rangle^{-n-1} * (\langle D\rangle^{-\tau}f)\|_{L^q(\mathbb{R}^n)} \\ &\leq C\|\langle D\rangle^{-\tau}f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

if we choose $\tau > n(\frac{1}{p}-\frac{1}{q})$. □

Define

$$\mathscr{E}_N f(x,t) = \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi+t|\xi|)}\mathcal{E}_{N,1}(t|\xi|)\hat{f}(\xi)\,\mathrm{d}\xi + \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi-t|\xi|)}\mathcal{E}_{N,2}(t|\xi|)\hat{f}(\xi)\,\mathrm{d}\xi.$$

Then we have the following lemma.

Lemma 2.2. *Let $q \geq p \geq 1$. There exists a constant $C > 0$ such that*

$$\left\| \sup_{t\in[1,2]} |\mathscr{E}_N f(\cdot,t)| \right\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \tag{2.10}$$

when

$$N > \max\left\{ 2\left(\frac{1}{q} - \operatorname{Re}\alpha\right), 2n\left(\frac{1}{p} - \frac{1}{q}\right) \right\}.$$

Proof. We fix a function φ as in (2.5). Let $\psi(r) = \varphi(r) - \varphi(2r)$ and $\psi_j(r) = \psi(2^{-j}r)$, for $j \geq 1$. So we have

$$1 \equiv \varphi(r) + \sum_{j\geq 1} \psi_j(r), \quad r \geq 0. \tag{2.11}$$

For $j \geq 1$, define

$$\mathscr{E}_{N,j} f(x,t) = \int_{\mathbb{R}^n} \left(e^{2\pi i(x\cdot\xi + t|\xi|)} \mathcal{E}_{N,1}(t|\xi|) + e^{2\pi i(x\cdot\xi - t|\xi|)} \mathcal{E}_{N,2}(t|\xi|) \right) \psi_j(t|\xi|) \hat{f}(\xi) d\xi.$$

To prove (2.10), it suffices to show that there exists a constant $\delta > 0$ such that for all $j \geq 1$,

$$\left\| \sup_{1\leq t\leq 2} |\mathscr{E}_{N,j} f(\cdot,t)| \right\|_{L^q(\mathbb{R}^n)} \leq C 2^{-\delta j} \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.12}$$

First, for each fixed $t \in [1,2]$, $f \to \mathscr{E}_{N,j} \langle D \rangle^{N/2} f(\cdot,t)$ is the sum of two Fourier integral operators of order $-(n-1)/2 - \operatorname{Re}\alpha - N/2$ with phase $x \cdot \xi \pm t|\xi|$. By [23, Theorem 2, Chapter IX] and the fact that $e^{it\sqrt{-\Delta}}$ is local at scale t, we have

$$\begin{aligned} \sup_{1\leq t\leq 2} \left\| \mathscr{E}_{N,j} f(\cdot,t) \right\|_{L^q(\mathbb{R}^n)} \leq & C 2^{-((n-1)/2 + \operatorname{Re}\alpha + N/2)j} 2^{(n-1)|1/2 - 1/q| j} \| \langle D \rangle^{-N/2} f \|_{L^q(\mathbb{R}^n)} \\ \leq & C 2^{-(\operatorname{Re}\alpha + N/2)j} \|f\|_{L^p(\mathbb{R}^n)}, \end{aligned} \tag{2.13}$$

if we choose $N > 2n\left(\frac{1}{p} - \frac{1}{q}\right)$.

Next, we write $\partial_t \mathscr{E}_{N,j} f(x,t)$ as the sum of following terms,

$$\begin{aligned} & \pm 2\pi i t^{-1} \int e^{2\pi i(x\cdot\xi \pm t|\xi|)} t|\xi| \mathcal{E}_{N,1}(t|\xi|) \psi_j(t|\xi|) \hat{f}(\xi) d\xi, \\ & \pm 2\pi i t^{-1} \int e^{2\pi i(x\cdot\xi \pm t|\xi|)} t|\xi| \mathcal{E}_{N,2}(t|\xi|) \psi_j(t|\xi|) \hat{f}(\xi) d\xi, \\ & t^{-1} \int e^{2\pi i(x\cdot\xi \pm t|\xi|)} t|\xi| (\mathcal{E}_{N,1}\psi_j)'(t|\xi|) \hat{f}(\xi) d\xi, \\ & t^{-1} \int e^{2\pi i(x\cdot\xi \pm t|\xi|)} t|\xi| (\mathcal{E}_{N,2}\psi_j)'(t|\xi|) \hat{f}(\xi) d\xi. \end{aligned}$$

By (2.4), we see that for each fixed $t \in [1,2]$, $f \to \partial_t \mathscr{E}_{N,j} f(\cdot,t)$ is the sum of Fourier integral operators of order no more than $-(n-1)/2 - \operatorname{Re}\alpha - N + 1$. By [23, Theorem 2, Chapter IX] again,

$$\sup_{1\le t\le 2} \left\| \partial_t \mathscr{E}_{N,j} f(\cdot,t) \right\|_{L^q(\mathbb{R}^n)} \le C 2^{-(\operatorname{Re}\alpha + N/2 - 1)j} \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.14}$$

With (2.13) and (2.14) at our disposal, we can apply [20, Lemma 2.4.2] to obtain

$$\left\| \sup_{1\le t\le 2} |\mathscr{E}_{N,j} f(\cdot,t)| \right\|_{L^q(\mathbb{R}^n)} \le C 2^{-(\operatorname{Re}\alpha + N/2 - 1/q)j} \|f\|_{L^p(\mathbb{R}^n)}.$$

Choosing $N > \max\{2(1/q - \operatorname{Re}\alpha), 2n(\frac{1}{p} - \frac{1}{q})\}$ and setting $\delta = \operatorname{Re}\alpha + N/2 - 1/q$, we obtain the desired estimate (2.12). The proof of Lemma 2.2 is complete. □

At the end of this section, we define

$$\mathscr{A}_t f(x) = \int_{\mathbb{R}^n} \Big(e^{2\pi i (x\cdot\xi + t|\xi|)} a_1(t|\xi|) + e^{2\pi i (x\cdot\xi - t|\xi|)} a_2(t|\xi|) \Big) \hat{f}(\xi)\, \mathrm{d}\xi. \tag{2.15}$$

From (2.5), Lemmas 2.1 and 2.2, we see that the $L^p \to L^q$ boundedness of the operator $\mathfrak{M}^\alpha$ reduces to the boundedness of the operator $\mathscr{A}_t$ on Sobolev spaces, which will be investigated in Section 3 below.

3 Proof of Theorem 1.1

To prove Theorem 1.1, we need to show the following Lemmas 3.1, 3.2 and 3.3.

Lemma 3.1. *Let $n \ge 2$ and $1 \le p, q \le \infty$. Suppose*

$$\left\| \mathfrak{M}_1^\alpha f \right\|_{L^q(\mathbb{R}^n)} \le C \|f\|_{L^p(\mathbb{R}^n)} \tag{3.1}$$

holds for some $\alpha \in \mathbb{C}$. Then, we have $q \ge p$ and

$$\operatorname{Re}\alpha \ge \frac{1}{p} - \frac{n}{q}.$$

Proof. Fix $N > \max\{2(\frac{1}{q} - \operatorname{Re}\alpha), 2n(\frac{1}{p} - \frac{1}{q})\}$. Let $\mathscr{A}_1$ be an operator given in (2.15). From (2.5), Lemma 2.1 and Lemma 2.2, we see that the proof of Lemma 3.1 reduces to showing the following result: Suppose

$$\|\mathscr{A}_1 f\|_{L^q(\mathbb{R}^n)} \le C \|f\|_{W^{s,p}(\mathbb{R}^n)} \tag{3.2}$$

holds for some $s \in \mathbb{R}$. Then we have $q \ge p$ and

$$s \ge \frac{n-1}{2} + \frac{1}{p} - \frac{n}{q}.$$

Since $\mathscr{A}_1$ is translation-invariant, an application of [7, Proposition 2.5.6] immediately yields $q \geq p$. Consider a nonnegative bump function $\phi \in C_c^\infty((0,\infty))$ satisfying $\phi = 1$ on $[1,2]$. For $j \geq 1$, define

$$\widehat{f_j}(\xi) = e^{-2\pi i|\xi|}\phi(2^{-j}|\xi|).$$

From [28, Lemma 2.1], we deduce the estimate

$$\|f_j\|_{L^p(\mathbb{R}^n)} \leq C2^{(\frac{n+1}{2}-\frac{1}{p})j}.$$

Then we have

$$\|\mathscr{A}_1 f_j\|_{L^q(\mathbb{R}^n)} \leq C\|f_j\|_{W^{s,p}(\mathbb{R}^n)} \leq C2^{sj}\|f_j\|_{L^p(\mathbb{R}^n)} \leq C2^{(s+\frac{n+1}{2}-\frac{1}{p})j}. \tag{3.3}$$

If j is large enough, we may apply (2.7) and (2.8) to obtain that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} a_1(|\xi|)\phi(2^{-j}|\xi|)\,d\xi\right| \geq& C^{-1}\int_{\mathbb{R}^n}\left|a_1(|\xi|)\phi(2^{-j}|\xi|)\right|\,d\xi\\ \geq& C^{-1}c_{low}\int_{\mathbb{R}^n}\left|\phi(2^{-j}|\xi|)\right|\,d\xi\\ \geq& C^{-1}c_{low}2^{nj}.\end{aligned}$$

Then by selecting $\varepsilon > 0$ sufficiently small, we use (2.11) to obtain that for all $|x| \leq \varepsilon 2^{-j}$,

$$\begin{aligned}\left|e^{2\pi i\sqrt{-\Delta}}a_1(|D|)f_j(x)\right| =& \left|\int_{\mathbb{R}^n} a_1(|\xi|)\phi(2^{-j}|\xi|)e^{-2\pi i x\cdot\xi}\,d\xi\right|\\ =& \left|\int_{\mathbb{R}^n} a_1(|\xi|)\phi(2^{-j}|\xi|)\,d\xi + \int_{\mathbb{R}^n} a_1(|\xi|)\phi(2^{-j}|\xi|)\left(e^{-2\pi i x\cdot\xi}-1\right)\,d\xi\right|\\ \geq& \left|\int_{\mathbb{R}^n} a_1(|\xi|)\phi(2^{-j}|\xi|)d\xi\right| - C\varepsilon\int\left|a_1(|\xi|)\phi(2^{-j}|\xi|)\right|\,d\xi\\ \geq& (2C)^{-1}\int\left|a_1(|\xi|)\phi(2^{-j}|\xi|)\right|\,d\xi \geq c2^{nj}\end{aligned} \tag{3.4}$$

for some $c > 0$. We can write

$$e^{-2\pi i\sqrt{-\Delta}}a_2(|D|)f_j(x) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi}e^{-4\pi i|\xi|}a_2(|\xi|)\phi(2^{-j}|\xi|)\,d\xi.$$

Note that the phase function $-2\pi x\cdot\xi - 4\pi|\xi|$ has no critical points when $|x| \leq \varepsilon$ and ε is small enough. So, by integration by parts, we have

$$\sup_{|x|\leq\varepsilon 2^{-j}} |e^{-2\pi i\sqrt{-\Delta}}a_2(|D|)f_j(x)| \leq C,$$

which, combined with (3.4), implies

$$\|\mathscr{A}_1 f_j\|_{L^q(\mathbb{R}^n)} \geq \|\mathscr{A}_1 f_j\|_{L^q(|x|\leq\varepsilon 2^{-j})} \geq C_\varepsilon 2^{(n-n/q)j}. \tag{3.5}$$

An application of both (3.3) and (3.5) yields

$$C_\varepsilon 2^{(n-n/q)j} \le C2^{(s+\frac{n+1}{2}-\frac{1}{p})j}.$$

Letting $j \to +\infty$, we conclude that $s \ge \frac{n-1}{2} + \frac{1}{p} - \frac{n}{q}$. □

Lemma 3.2. *Let $n \ge 2$, $1 \le p,q \le \infty$. Suppose*

$$\Big\| \sup_{1\le t\le 2} |\mathfrak{M}_t^\alpha f| \Big\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \tag{3.6}$$

holds for some $\alpha \in \mathbb{C}$. Then, we have $q \ge p$ and

$$\operatorname{Re}\alpha \ge \frac{n+1}{2p} - \frac{n-1}{2q} - \frac{n-1}{2}.$$

Proof. Fix $N > \max\{2(\frac{1}{q} - \operatorname{Re}\alpha), 2n(\frac{1}{p} - \frac{1}{q})\}$ as in Lemma 2.2. From (2.5), Lemma 2.1 and Lemma 2.2, we see that the proof of Lemma 3.2 reduces to showing the following: Suppose

$$\Big\| \sup_{1\le t\le 2} |\mathscr{A}_t f| \Big\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{W^{s,p}(\mathbb{R}^n)} \tag{3.7}$$

holds for some $s \in \mathbb{R}$. Then we have $q \ge p$ and $s \ge \frac{n+1}{2p} - \frac{n-1}{2q}$.

Note that (3.7) implies that $\|\mathscr{A}_1 f\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{W^{s,p}(\mathbb{R}^n)}$. This fact, together with the translation-invariant property of $\mathscr{A}_1$, yields that $q \ge p$. We now prove that $s \ge \frac{n+1}{2p} - \frac{n-1}{2q}$. Denote $\xi = (\xi_1, \xi') \in \mathbb{R}^n$. For $j \ge 1$ and $\delta > 0$, we let $\hat{f} \ge 0$ be a smooth cut-off function of the set

$$\Big\{(\xi_1,\xi') \in \mathbb{R}^n : |\xi_1 - 2^j| \le \delta 2^{j-1},\ |\xi'| \le \delta 2^{j/2}\Big\}$$

such that

$$\big|\partial_\xi^\beta \hat{f}(\xi)\big| \le C_{\delta,\beta} 2^{-j|\beta'|/2} 2^{-j|\beta_1|}$$

for any $\beta = (\beta_1, \beta') \in \mathbb{Z}_+^n$.

It follows from (3.22) of [11] that, if j is large enough and δ is small enough, we have

$$\sup_{1\le t\le 2} |\mathscr{A}_t f| \ge C^{-1}\delta^n 2^{\frac{n+1}{2}j} \tag{3.8}$$

for all $1 \le x_1 \le 2$, $|x'| \le 2^{-j/2}$. Then we have

$$\begin{aligned} \Big\| \sup_{1\le t\le 2} |\mathscr{A}_t f| \Big\|_{L^q(\mathbb{R}^n)} &\ge \| \sup_{1\le t\le 2} |\mathscr{A}_t f| \|_{L^q(1\le x_1\le 2, |x'|\le 2^{-j/2})} \\ &\ge C^{-1}\delta^n 2^{\frac{n+1}{2}j} 2^{-\frac{n-1}{2q}j}. \end{aligned} \tag{3.9}$$

By (3.7) and the definition of f, we have

$$\Big\| \sup_{1\le t\le 2} |\mathscr{A}_t f| \Big\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{W^{s,p}} \le C_\delta 2^{sj} 2^{\frac{n+1}{2}j-\frac{n+1}{2p}j}. \tag{3.10}$$

Combining (3.9) and (3.10), we obtain

$$C^{-1}\delta^n 2^{\frac{n+1}{2}j} 2^{-\frac{n-1}{2q}j} \le C_\delta 2^{sj} 2^{\frac{n+1}{2}j-\frac{n+1}{2p}j}.$$

Passing to the limit $j\to+\infty$ yields $\frac{n+1}{2}-\frac{n-1}{2q}\le s+\frac{n+1}{2}-\frac{n+1}{2p}$, which means $s\ge\frac{n+1}{2p}-\frac{n-1}{2q}$. □

Lemma 3.3. *Let $n\ge 2$, $1\le p,q\le\infty$. Suppose*

$$\Big\| \sup_{1\le t\le 2} |\mathfrak{M}_t^\alpha f| \Big\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \tag{3.11}$$

holds for some $\alpha\in\mathbb{C}$. Then, we have $q\ge p$ and

$$\mathrm{Re}\,\alpha \ge \frac{n}{p}-n+1.$$

Proof. Fix $N>\max\{2(\frac{1}{q}-\mathrm{Re}\,\alpha),2n(\frac{1}{p}-\frac{1}{q})\}$ as in Lemma 2.2. By employing (2.5) together with Lemmas 2.1 and 2.2, Lemma 3.3 reduces to showing the following result: If

$$\Big\| \sup_{1\le t\le 2} |\mathscr{A}_t f| \Big\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{W^{s,p}(\mathbb{R}^n)} \tag{3.12}$$

holds for some $s\in\mathbb{R}$, then we have $q\ge p$ and $s\ge\frac{n}{p}-\frac{n-1}{2}$.

Since $\mathscr{A}_1$ is translation-invariant, we have that $q\ge p$. We now establish the necessary condition $s\ge\frac{n}{p}-\frac{n-1}{2}$. Assume that $\chi(\xi)\in C^\infty(\mathbb{R}^n\backslash\{0\})$ is homogeneous of order 0 satisfying $\chi(\xi)=1$ if $|\frac{\xi}{|\xi|}-v_1|\le 10^{-2}$ and vanishes if $|\frac{\xi}{|\xi|}-v_1|\ge 9^{-2}$, where $v_1:=(1,0,\cdots,0)$. Let $\hat{f}_j(\xi):=\phi(2^{-j}|\xi|)\chi(\xi)$, where $\phi\in C_c^\infty((0,\infty))$, $\phi=1$ on $[1,2]$ and $\phi\ge 0$.

Note that

$$e^{2\pi it\sqrt{-\Delta}}a_1(t|D|)f_j(x)=\int e^{2\pi ix\cdot\xi}e^{2\pi it|\xi|}a_1(t|\xi|)\phi(2^{-j}|\xi|)\chi(\xi)\,d\xi.$$

If $|\frac{x}{|x|}-v_1|\le 10^{-2}$, the phase function of the above integral has no critical points and thus

$$\sup_{t\in[1,2]} |e^{2\pi it\sqrt{-\Delta}}a_1(t|D|)f_j(x)| \le C2^{-nj}. \tag{3.13}$$

It is known from [23, p. 360] that for $|x| \geq 1$ and $|\frac{x}{|x|} - v_1| \leq 9^{-2}$, there holds

$$\widehat{\chi \mathrm{d}\sigma}(-x) = e^{2\pi i|x|}h(-x) + e(-x),$$

where e belongs to $S^{-\infty}$ and $h \in S^{-(n-1)/2}$ can be splited into two terms:

$$h(x) = c_0|x|^{-(n-1)/2}\chi(-x/|x|) + \tilde{e}(x), \quad \tilde{e} \in S^{-(n+1)/2},$$

for all $|x| \geq 1$. Here, for each $m \in \mathbb{R}$, we denote by S^m the standard symbol class as defined in [23, Chapter VI, Section 1.3], and we set $S^{-\infty} = \cap_{m\in\mathbb{R}} S^m$. Then for all $|x| \geq 1$, $|\frac{x}{|x|} - v_1| \leq 9^{-2}$ and $j \geq 1$, we have

$$\begin{aligned}
&e^{-2\pi it\sqrt{-\Delta}}a_2(t|D|)f_j(x)\\
=&\int e^{2\pi ix\cdot\xi}e^{-2\pi it|\xi|}a_2(t|\xi|)\phi(2^{-j}|\xi|)\chi(\xi)\,d\xi\\
=&\int_0^\infty (\chi d\sigma)^\wedge(-rx)e^{-2\pi itr}a_2(tr)\phi(2^{-j}r)r^{n-1}\,dr\\
=&c_0|x|^{-\frac{n-1}{2}}\int_0^\infty e^{2\pi ir(|x|-t)}\chi(x/|x|)a_2(tr)\phi(2^{-j}r)r^{\frac{n-1}{2}}\,\mathrm{d}r + R_j(x,t),
\end{aligned}$$

where

$$\begin{aligned}
R_j(x,t) :=& c_1\int e^{2\pi ir(|x|-t)}\tilde{e}(-rx)a_2(tr)\phi(2^{-j}r)r^{n-1}\,dr\\
&+ c_2\int e^{-2\pi itr}e(-rx)a_2(tr)\phi(2^{-j}r)r^{n-1}\,dr
\end{aligned}$$

and $c_0, c_1, c_2 > 0$. Let j be large enough. Then if $|\frac{x}{|x|} - v_1| \leq 10^{-2}$ and $\big||x| - t\big| \leq \delta 2^{-j}$ for some sufficiently small $\delta \in (0,1)$, by (2.7) and (2.8), we have

$$\left|e^{-2\pi it\sqrt{-\Delta}}a_2(t|D|)f_j(x)\right| \geq c\left|\int \phi(2^{-j}r)r^{\frac{n-1}{2}}\,dr\right| - C2^{\frac{n-1}{2}j} \geq \tilde{c}2^{\frac{n+1}{2}j}$$

for some positive constants c and $\tilde{c}$. Then if $|x| \in [1,2]$ and $\left|\frac{x}{|x|} - v_1\right| \leq 10^{-2}$,

$$\sup_{t\in[1,2]}\left|e^{-2\pi it\sqrt{-\Delta}}a_2(t|D|)f_j(x)\right| \geq \tilde{c}2^{\frac{n+1}{2}j},$$

which, combined with (3.13), yields

$$\sup_{t\in[1,2]}|\mathscr{A}_t f_j| \geq \sup_{t\in[1,2]}\left|e^{-2\pi it\sqrt{-\Delta}}a_2(t|D|)f_j\right| - \sup_{t\in[1,2]}\left|e^{2\pi it\sqrt{-\Delta}}a_1(t|D|)f_j\right| \geq \frac{\tilde{c}}{2}2^{\frac{n+1}{2}j}.$$

Hence,

$$\left\|\sup_{t\in[1,2]}|\mathscr{A}_t f_j|\right\|_{L^q(\mathbb{R}^n)} \geq \|\sup_{t\in[1,2]}|\mathscr{A}_t f_j|\|_{L^q(|x|\in[1,2],|\frac{x}{|x|}-v_1|\leq 10^{-2})} \geq C^{-1}\frac{\tilde{c}}{2}2^{\frac{n+1}{2}j}. \tag{3.14}$$

On the other hand, choose $\tilde{\phi} \in C_c^\infty((0,\infty))$ satisfying $\tilde{\phi} = 1$ on $\operatorname{supp}\phi$, then $\hat{f}_j(\xi) = \phi(2^{-j}|\xi|)\tilde{\phi}(2^{-j}|\xi|)\chi(\xi)$. Since $\tilde{\phi}(2^{-j}|D|)\chi(D)$ is bounded on $L^p(\mathbb{R}^n)$, we have

$$\Big\| \sup_{t\in[1,2]} |\mathscr{A}_t f_j| \Big\|_{L^q(\mathbb{R}^n)} \leq C\|f_j\|_{W^{s,p}(\mathbb{R}^n)} \leq C2^{sj}2^{nj}2^{-\frac{n}{p}j}. \tag{3.15}$$

By (3.14) and (3.15), we conclude that

$$C^{-1}\frac{\tilde{c}}{2}2^{\frac{n+1}{2}j} \leq C2^{sj}2^{nj}2^{-\frac{n}{p}j}.$$

Letting $j \to +\infty$, we obtain $s \geq \frac{n}{p} - \frac{n-1}{2}$. □

We finally present the endgame in the proof of Theorem 1.1.

Proof of Theorem 1.1. This is a consequence of Lemmas 3.1, 3.2 and 3.3. □

4 Proof of Theorem 1.2

To prove Theorem 1.2, we will first give the $L^p \to L^q$ local smoothing estimates for the wave operator $e^{it\sqrt{-\Delta}}$, i.e., Theorem 4.1 for $n=2$ and Theorem 4.2 for $n>2$ below.

Theorem 4.1. *Let $1 \leq p \leq q \leq \infty$. Define $s_2(p,q)$ as follows:*

$$s_2(p,q) = \begin{cases} \dfrac{1}{2} + \dfrac{1}{p} - \dfrac{3}{q} & \text{for} \quad q \geq 3p', \\ \dfrac{3}{2p} - \dfrac{3}{2q} & \text{for} \quad p' \leq q < 3p', \\ \dfrac{2}{p} - \dfrac{1}{2} - \dfrac{1}{q} & \text{for} \quad q < p', \end{cases} \tag{4.1}$$

where $p' := p/(p-1)$. If $s > s_2(p,q)$, then we have

$$\left(\int_1^2 \|e^{it\sqrt{-\Delta}} f\|_{L^q(\mathbb{R}^2)}^q dt \right)^{1/q} \leq C_s \|f\|_{W^{s,p}(\mathbb{R}^2)}. \tag{4.2}$$

Proof. Let ε be any positive number. By the local smoothing estimate of Guth,Wang and Zhang [8], we have the following $(4,4)$ estimate:

$$\left(\int_1^2 \|e^{it\sqrt{-\Delta}} f\|_{L^4(\mathbb{R}^2)}^4 \, dt \right)^{1/4} \leq C_\varepsilon \|f\|_{W^{\varepsilon,4}(\mathbb{R}^2)}. \tag{4.3}$$

It is also known that the fixed-time estimate of Seeger, Sogge and Stein [19] implies the following $(1,1)$ and (∞,∞) estimates:

$$\|e^{it\sqrt{-\Delta}} f\|_{L^p(\mathbb{R}^2\times[1,2])} \leq C_\varepsilon \|f\|_{W^{\frac{1}{2}+\varepsilon,p}(\mathbb{R}^2)}, \tag{4.4}$$

where $p = 1$ or ∞. Moreover, it follows from [23, Chapter IX, 6.16] that the following $(1,\infty)$ estimate holds

$$\sup_{t\in[1,2]} \|e^{it\sqrt{-\Delta}}f\|_{L^\infty(\mathbb{R}^2)} \le C_\varepsilon \|f\|_{W^{\frac{3}{2}+\varepsilon,1}(\mathbb{R}^2)}. \tag{4.5}$$

Theorem 4.1 follows by interpolating between the key estimates (4.3)–(4.5). More precisely, in the case $q \ge 3p'$, (4.2) follows from the interpolation between (∞,∞), $(4,4)$ and $(1,\infty)$ estimates. In the case $p' < q \le 3p'$, (4.2) follows from the interpolation between $(4,4)$, $(2,2)$ and $(1,\infty)$ estimates. In the case $q < p'$, (4.2) follows from the interpolation between $(2,2)$, $(1,1)$ and $(1,\infty)$ estimates. □

The following estimate is proven in [4, Proposition 2.1].

Lemma 4.1. *Let $n \ge 3$ and suppose $p_0 = \frac{2(n^2+2n-1)}{(n-1)(n+3)}$, $q_0 = \frac{2(n^2+2n-1)}{(n-1)(n+1)}$, $s_0 = \frac{(n-1)(n+1)}{2(n^2+2n-1)}$. Then*

$$\left(\int_1^2 \|e^{it\sqrt{-\Delta}}f\|^{q_0}_{L^{q_0}(\mathbb{R}^n)} dt\right)^{1/q_0} \le C2^{s_0 k}\|f\|_{L^{p_0}(\mathbb{R}^n)}$$

holds for all $k \ge 1$ and $f \in \mathcal{S}'$ with $\operatorname{supp}\hat{f} \subseteq \{\xi \in \mathbb{R}^n :\ 2^{k-1} \le |\xi| \le 2^{k+1}\}$.

Recall that O, A, B, C, D, E are given in Fig. 1 in Section 1.

Theorem 4.2. *Suppose $n \ge 3$. Let $1 \le p \le q \le \infty$. Define $s_n(p,q)$ as follows:*

$$s_n(p,q) = \begin{cases} \dfrac{1}{p} - \dfrac{n+1}{q} + \dfrac{n-1}{2} & \text{for } \left(\dfrac{1}{p}, \dfrac{1}{q}\right) \in \Delta AOE \cup \Delta ABE, \\ \dfrac{n+1}{2}\left(\dfrac{1}{p} - \dfrac{1}{q}\right) & \text{for } \left(\dfrac{1}{p}, \dfrac{1}{q}\right) \in \Delta BCE, \\ \dfrac{1}{2p} - \dfrac{n}{2q} + \dfrac{n-1}{4} & \text{for } \left(\dfrac{1}{p}, \dfrac{1}{q}\right) \in \Delta ABC, \\ \dfrac{n}{p} - \dfrac{1}{q} - \dfrac{n-1}{2} & \text{for } \left(\dfrac{1}{p}, \dfrac{1}{q}\right) \in \Delta CDE. \end{cases}$$

If $s > s_n(p,q)$, we have

$$\left(\int_1^2 \|e^{it\sqrt{-\Delta}}f\|^{q}_{L^{q}(\mathbb{R}^n)} dt\right)^{1/q} \le C_s\|f\|_{W^{s,p}(\mathbb{R}^n)}. \tag{4.6}$$

Proof. Let ε be any positive number. By applying the decoupling inequality of Bourgain and Demeter [3], we have that for all $p \ge \frac{2n+2}{n-1}$,

$$\left(\int_1^2 \|e^{it\sqrt{-\Delta}}f\|_{L^{p}(\mathbb{R}^n)}\right)^{1/p} \le C_\varepsilon\|f\|_{W^{\frac{n-1}{2}-\frac{n}{p}+\varepsilon,p}(\mathbb{R}^n)}.$$

The fixed-time estimate established by Seeger, Sogge, and Stein [19] yields

$$\|e^{it\sqrt{-\Delta}}f\|_{L^p(\mathbb{R}^n\times[1,2])}\le C_\varepsilon\|f\|_{W^{\frac{n-1}{2}+\varepsilon,p}(\mathbb{R}^n)},$$

where $p=1$ or ∞. By [23, Chapter IX, 6.16], we have the following $(1,\infty)$ estimate:

$$\sup_{t\in[1,2]}\|e^{it\sqrt{-\Delta}}f\|_{L^\infty(\mathbb{R}^n)}\le C_\varepsilon\|f\|_{W^{\frac{n+1}{2}+\varepsilon,1}(\mathbb{R}^n)}.$$

Combining Lemma 4.1 with a frequency decomposition argument, we obtain the following (p_0,q_0) estimate:

$$\left(\int_1^2\|e^{it\sqrt{-\Delta}}f\|^{q_0}_{L^{q_0}(\mathbb{R}^n)}dt\right)^{1/q_0}\le C_\varepsilon\|f\|_{W^{s_0+\varepsilon,p_0}(\mathbb{R}^n)}.$$

With these estimates at our disposal, Theorem 4.2 can be proved by interpolation. More precisely, in the case $(\frac{1}{p},\frac{1}{q})\in\Delta AOE\cup\Delta ABE$, (4.6) follows from the interpolation between (∞,∞), $(\frac{2n+2}{n-1},\frac{2n+2}{n-1})$, (p_0,q_0) and $(1,\infty)$ estimates. In the case $(\frac{1}{p},\frac{1}{q})\in\Delta BCE$, (4.6) follows from the interpolation between (p_0,q_0), $(2,2)$ and $(1,\infty)$ estimates. In the case $(\frac{1}{p},\frac{1}{q})\in\Delta ABC$, (4.6) follows from the interpolation between $(2,2)$, (p_0,q_0) and $(\frac{2n+2}{n-1},\frac{2n+2}{n-1})$ estimates. In the case $(\frac{1}{p},\frac{1}{q})\in\Delta CDE$, (4.6) follows from the interpolation between $(2,2)$, $(1,1)$ and $(1,\infty)$ estimates. □

Lemma 4.2. *Suppose that $g(x,\cdot)\in C^1([1,2])$ for a.e. $x\in\mathbb{R}^n$. Then, the following estimates hold:*

(i) *For $1<q<\infty$,*

$$\left\|\sup_{t\in[1,2]}|g(\cdot,t)|\right\|_{L^q(\mathbb{R}^n)}\le C\|g\|_{L^q(\mathbb{R}^n\times[1,2])}+C\|g\|^{1-\frac{1}{q}}_{L^q(\mathbb{R}^n\times[1,2])}\|\partial_t g\|^{\frac{1}{q}}_{L^q(\mathbb{R}^n\times[1,2])}.$$

(ii) *For $q=1$,*

$$\left\|\sup_{t\in[1,2]}|g(\cdot,t)|\right\|_{L^1(\mathbb{R}^n)}\le C\|g\|_{L^1(\mathbb{R}^n\times[1,2])}+C\|\partial_t g\|_{L^1(\mathbb{R}^n\times[1,2])}.$$

Proof. For $1<q<\infty$, the proof is a slight variant of [20, Lemma 2.4.2]. For a.e. $x\in\mathbb{R}^n$ and each $t_0\in[1,2]$, we have

$$\sup_{t\in[1,2]}|g(x,t)|^q\le|g(x,t_0)|^q+q\int_1^2|g(x,s)|^{q-1}|\partial_s g(x,s)|\,ds.$$

Integrating with respect to x and applying Hölder's inequality yields

$$\begin{aligned}\Big\|\sup_{t\in[1,2]}|g(\cdot,t)|\Big\|_{L^q(\mathbb{R}^n)}\leq& C\inf_{t\in[1,2]}\|g(\cdot,t)\|_{L^q(\mathbb{R}^n)}+C\|g\|^{1-\frac{1}{q}}_{L^q(\mathbb{R}^n\times[1,2])}\|\partial_t g\|^{\frac{1}{q}}_{L^q(\mathbb{R}^n\times[1,2])}\\ \leq& C\|g\|_{L^q(\mathbb{R}^n\times[1,2])}+C\|g\|^{1-\frac{1}{q}}_{L^q(\mathbb{R}^n\times[1,2])}\|\partial_t g\|^{\frac{1}{q}}_{L^q(\mathbb{R}^n\times[1,2])}.\end{aligned}$$

The case $q=1$ follows by an analogous argument, which we omit here. □

Proposition 4.1. *Let $n\geq 2$ and $1\leq p\leq q\leq\infty$. If the local smoothing estimate*

$$\|e^{it\sqrt{-\Delta}}f\|_{L^q(\mathbb{R}^n\times[1,2])}\leq C_{n,p,q,s}\|f\|_{W^{s,p}(\mathbb{R}^n)} \tag{4.7}$$

holds for some $s\in\mathbb{R}$, then we have

$$\Big\|\sup_{t\in[1,2]}|\mathfrak{M}^\alpha_t f|\Big\|_{L^q(\mathbb{R}^n)}\leq C_{n,p,q,\alpha}\|f\|_{L^p(\mathbb{R}^n)} \tag{4.8}$$

whenever $\mathrm{Re}\,\alpha>s-\frac{n-1}{2}+\frac{1}{q}$.

Proof. Let φ and $\{\psi_j\}_{j\geq1}$ be functions as in (2.11). We write

$$\begin{aligned}\widehat{\mathfrak{M}^\alpha_t f}(\xi)=&\varphi(t|\xi|)\widehat{m_\alpha}(t\xi)\hat{f}(\xi)+\sum_{j\geq1}\psi_j(t|\xi|)\widehat{m_\alpha}(t\xi)\hat{f}(\xi)\\ =:&\widehat{\mathfrak{M}^\alpha_{0,t}f}(\xi)+\sum_{j\geq1}\widehat{\mathfrak{M}^\alpha_{j,t}f}(\xi).\end{aligned} \tag{4.9}$$

By applying (4.9) and Lemma 2.1, the proof of (4.8) reduces to showing that there exists $\delta>0$ such that for all $j\geq1$, there holds

$$\Big\|\sup_{t\in[1,2]}|\mathfrak{M}^\alpha_{j,t}f|\Big\|_{L^q(\mathbb{R}^n)}\leq C2^{-\delta j}\|f\|_{L^p(\mathbb{R}^n)} \tag{4.10}$$

whenever $\mathrm{Re}\,\alpha>s-\frac{n-1}{2}+\frac{1}{q}$.

By (2.5), together with Lemma 2.2, we reduce (4.10) to

$$\Big\|\sup_{t\in[1,2]}|\mathscr{A}_{j,t}f|\Big\|_{L^q(\mathbb{R}^n)}\leq C2^{(s+\frac{1}{q})j}\|f\|_{L^p(\mathbb{R}^n)}, \tag{4.11}$$

where $\widehat{\mathscr{A}_{j,t}f}(\xi):=\psi_j(t|\xi|)\widehat{\mathscr{A}_t f}(\xi)$ and $\mathscr{A}_t f$ is defined as in (2.15). By (2.6), we can write

$$\mathscr{A}_{j,t}f(x)=C\sum_{\ell=0}^{N-1}\int_{\mathbb{R}^n}\Big(b_\ell e^{2\pi i(x\cdot\xi+t|\xi|)}+d_\ell e^{2\pi i(x\cdot\xi-t|\xi|)}\Big)|t\xi|^{-\ell}\psi_j(t|\xi|)\hat{f}(\xi)\,\mathrm{d}\xi,$$

which is a linear combination of

$$T_{\ell,j}f(x,t) := \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi \pm t|\xi|)}|t\xi|^{-\ell}\psi_j(t|\xi|)\hat{f}(\xi)\,\mathrm{d}\xi, \quad \ell = 0,1,\cdots,N-1.$$

Hence, the proof of (4.11) reduces to showing that

$$\Big\|\sup_{t\in[1,2]}|T_{0,j}f(\cdot,t)|\Big\|_{L^q(\mathbb{R}^n)} \le C2^{(s+\frac{1}{q})j}\|f\|_{L^p(\mathbb{R}^n)}, \quad j\ge 1. \tag{4.12}$$

We observe that

$$\sup_{j\ge 1}\sup_{t\in[1,2]}\big|\partial_\xi^\beta\big(\psi_j(t|\xi|)\big)\big| \le C_\beta(1+|\xi|)^{-|\beta|},$$

where β is any multi-index. Consequently, $\psi_j(t|\cdot|)\in S^0$ uniformly for all $1\le t\le 2$ and $j\ge 1$, which implies the existence of a constant $C>0$, independent of both t and j, such that

$$\begin{aligned}&\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi\pm t|\xi|)}\psi_j(t|\xi|)\hat{f}(\xi)\,\mathrm{d}\xi\right|^p \mathrm{d}x\\ \le& C\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi\pm t|\xi|)}\tilde{\psi}_j(\xi)\hat{f}(\xi)\,\mathrm{d}\xi\right|^p \mathrm{d}x.\end{aligned} \tag{4.13}$$

Here $\tilde{\psi}_j(\xi)$ equals to 1 if $|\xi|\in[2^{j-2}M,2^{j+1}M]$ and vanishes if $|\xi|\notin[2^{j-3}M,2^{j+2}M]$, so that $\tilde{\psi}_j$ equals to 1 on the support of $\psi_j(t|\cdot|)$ when $1\le t\le 2$. By applying the assumption (4.7) to (4.13), we have

$$\|T_{0,j}f\|_{L^q(\mathbb{R}^n\times[1/2,2])} \le C2^{sj}\|f\|_{L^p(\mathbb{R}^n)}. \tag{4.14}$$

By the same token, the operator

$$\partial_t T_{0,j}f(x,t) = \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi\pm t|\xi|)}\big(\pm 2\pi i|\xi|\psi_j(t|\xi|) + |\xi|(\psi_j)'(t|\xi|)\big)\hat{f}(\xi)\,\mathrm{d}\xi$$

satisfies

$$\|\partial_t T_{0,j}f\|_{L^q(\mathbb{R}^n\times[1/2,2])} \le C2^{(s+1)j}\|f\|_{L^p(\mathbb{R}^n)}. \tag{4.15}$$

With (4.14) and (4.15) at our disposal, (4.12) follows from Lemma 4.2 when $q<\infty$, while the case $q=\infty$ is immediate from (4.14). □

Proof of Theorem 1.2. By noting that $\sigma_2(p,q) = s_2(p,q) - \frac{1}{2} + \frac{1}{q}$ and $d_n(p,q) = s_n(p,q) - \frac{n-1}{2} + \frac{1}{q}$ for $n>2$, Theorem 1.2 is a direct consequence of Proposition 4.1 and Theorems 4.1, 4.2. □

Remark 4.1. In the dimension $n \geq 3$ Gao et al. [5] obtained improved local smoothing estimates for the wave equation, that is,

$$\|e^{it\sqrt{-\Delta}}f\|_{L^p(\mathbb{R}^n\times[1,2])} \leq C_{n,p}\|f\|_{W^{s,p}(\mathbb{R}^n)}$$

holds with $s=(n-1)(1/2-1/p)-\sigma$ for all $\sigma<2/p-1/2$ when

$$p > \begin{cases} \dfrac{2(3n+5)}{3n+1} & \text{for } n \text{ odd,} \\ \dfrac{2(3n+6)}{3n+2} & \text{for } n \text{ even.} \end{cases}$$

Applying Proposition 4.1, we can improve (ii) of Theorem 1.2. However, the range of α is not optimal. What happens when $n \geq 3$ remains open.

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References

[1] D. Beltran, J. Hickman, and C. D. Sogge, Variable coefficient Wolff-type inequalities and sharp local smoothing estimates for wave equations on manifolds, Anal. PDE, 13(2) (2020), 403–433.

[2] J. Bourgain, Averages in the plane over convex curves and maximal operators, J. Anal. Math., 47 (1986), 69–85.

[3] J. Bourgain and C. Demeter, The proof of the l^2 decoupling conjecture, Ann. Math. (2), 182(1) (2015), 351–389.

[4] C. Cho, S. Lee and W. Li, Endpoint estimates for maximal operators associated to the wave equation, J. Funct. Anal., 290(6) (2026), Paper No. 111308.

[5] C.W. Gao, B.X. Liu, C.X. Miao, and Y.K.Xi. Improved local smoothing estimate for the equation in higher dimensions, J. Funct. Anal., 284 (2023), Paper No. 109879.

[6] G. Garrigs and A. Seeger, A mixed norm variant of Wolff's inequality for paraboloids, In Harmonic analysis and partial differential equations, volume 505 of Contemp. Math., pp. 179–197. American Mathematical Society, Providence (2010).

[7] L. Grafakos, Classical Fourier Analysis, Third Edition, Graduate Texts in Mathematics, 249. Springer, New York, 2014.

[8] L. Guth, H. Wang, and R.X. Zhang, A sharp square function estimate for the cone in $\mathbb{R}^3$, Ann. Math. (2), 192 (2020), 551–581.

[9] I. Laba and T. Wolff, A local smoothing estimate in higher dimensions, J. Anal. Math., 88 (2002), 149–171.
[10] S. Lee, Endpoint estimates for the circular maximal function, Proc. Amer. Math. Soc., 131 (2003), 1433–1442.
[11] N. J. Liu, M. X. Shen, L. Song and L. X. Yan. L^p bounds for Stein's spherical maximal operators, Math. Ann., 390 (2024), 5235–5255.
[12] C. X. Miao, J. W. Yang, and J. Q. Zheng, On local smoothing problems and Stein's maximal spherical means, Proc. Amer. Math. Soc., 145 (2017), 4269–4282.
[13] G. Mockenhaupt, A. Seeger, and C. D. Sogge, Wave front sets, local smoothing and Bourgain's circular maximal theorem, Ann. Math. (2), 136 (1992), 207–218.
[14] G. Mockenhaupt, A. Seeger, and C. D. Sogge, Local smoothing of Fourier integral operators and Carleson-Sjölin estimates, J. Amer. Math. Soc., 6 (1993), 65–130.
[15] A. Nowak, L. Roncal, and T. Z. Szarek, Endpoint estimates and optimality for the generalized spherical maximal operator on radial functions, Commun. Pure Appl. Anal., 22 (2023), 2233–2277.
[16] W. Schlag, $L^p \to L^q$ Estimates for the Circular Maximal Function, Ph.D. Thesis. California Institute of Technology, (1996).
[17] W. Schlag, A generalization of Bourgain's circular maximal theorem, J. Am. Math. Soc., 10 (1997), 103–122.
[18] W. Schlag and C. Sogge, Local smoothing estimates related to the circular maximal theorem, Math. Res. Lett., 4 (1997), 1–15.
[19] A. Seeger, C. D. Sogge, and E. M. Stein, Regularity properties of Fourier integral operators, Ann. Math. (2), 134 (1991), 231–251.
[20] C. Sogge, Fourier integrals in Classical Analysis, Second edition, Cambridge Tracts in Mathematics, 210. Cambridge University Press, Cambridge, 2017. xiv+334 pp.
[21] C. Sogge, Propagation of singularities and maximal functions in the plane, Invent. Math., 104 (1991), 349–376.
[22] E. M. Stein, Maximal functions, Spherical means, Proc. Nat. Acad. Sci. U.S.A., 73 (1976), 2174–2175.
[23] E. M. Stein, Harmonic analysis: Real variable methods, orthogonality and oscillatory integrals. With the assistance of Timothy S. Murphy. Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.
[24] E. M. Stein and S. Wainger, Problems in harmonic analysis related to curvature, Bull. Amer. Math. Soc., 84 (1987), 1239–1295.
[25] E. M. Stein and G. Weiss, Introduction to Fourier analysis on Euclidean spaces, Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.
[26] G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge; The Macmillan Company, New York, 1944.
[27] T. Wolff, Local smoothing type estimates on L^p for large p, Geom. Funct. Anal., 10 (2000), 1237–1288.
[28] F. Zhang, A note on generalized shperical maximal means, arXiv:2404.10437.