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Bilinear Fractional Integral Operators

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Dedicated to the memory of Prof. Donggao Deng on the occasion of his 90th birthday

Abstract. We study the bilinear fractional integral considered by Kenig and Stein, where linear combinations of variables with matrix coefficients are involved. Under more general settings, we give a complete characterization of the corresponding parameters for which the bilinear fractional integral is bounded from $L^{p_1}(\mathbb{R}^{n_1}) \times L^{p_2}(\mathbb{R}^{n_2})$ to $L^q(\mathbb{R}^m)$.

Key Words: Bilinear fractional integrals, Riesz potentials.
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1 Introduction and the main result

The multilinear singular integral operators have been widely studied since Coifman and Meyer's pioneer work [8]. Christ and Journé [7], Grafakos and Torres [15] developed the theory of multilinear Calderón-Zygmund operators. Lacey and Thiele [20,21] proved the boundedness of the bilinear Hilbert transform.

In this paper, we focus on the bilinear fractional integral studied by Kenig and Stein [17], Grafakos and Kalton [10, 13], Grafakos and Lynch [14]. Recall that for $0 < \lambda < 2n$, $f_1 \in L^{p_1}(\mathbb{R}^n)$ and $f_2 \in L^{p_2}(\mathbb{R}^n)$, the bilinear fractional integral of (f_1, f_2) is defined by

$$\int_{\mathbb{R}^{2n}} \frac{f_1(y_1) f_2(y_2) dy_1 dy_2}{(|x-y_1| + |x-y_2|)^\lambda}.$$

It was shown in [17, Lemma 7] that the bilinear fractional integral is bounded from $L^{p_1} \times L^{p_2}$ to L^q when the indices satisfy certain conditions. We refer to [1–4, 9, 16, 19, 22–24] for some recent advances on the study of bilinear fractional integrals and their applications in PDE. See also [5,6] for a generalization to mixed-norm Lebesgue spaces.

Komori-Furuya [18] gave a complete characterization of the indices for which the multilinear fractional integral is bounded. Here we cite a bilinear version.

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Proposition 1.1 ([18]). *Suppose that* $1 \leq p_1, p_2 \leq \infty$, $0 < \lambda < 2n$ *and* $1/p_1 + 1/p_2 = 1/q + (2n-\lambda)/n$. *Then the norm estimate*

$$\left\| \int_{\mathbb{R}^{2n}} \frac{f_1(y_1) f_2(y_2) dy_1 dy_2}{(|x-y_1| + |x-y_2|)^\lambda} \right\|_{L^q_x} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}$$

is true if and only if

(a) *either* $1 < p_1 < \infty$ *or* $1 < p_2 < \infty$,

(b) *the index* q *satisfies*

$$\begin{cases} \max\{p_1, p_2\} \leq q < \infty, & \text{if } \min\{p_1,p_2\} = 1, \\ \min\{p_1,p_2\} < q < \infty, & \text{if } \max\{p_1,p_2\} = \infty, \\ 0 < 1/q < 1/p_1 + 1/p_2, & \text{if } 1 < p_1, p_2 < \infty \text{ and } 1/p_1 + 1/p_2 < 1, \\ 0 \leq 1/q < 1/p_1 + 1/p_2, & \text{if } 1 < p_1, p_2 < \infty \text{ and } 1/p_1 + 1/p_2 \geq 1. \end{cases}$$

Kenig and Stein [17, Remark 10] studied the multi-linear fractional integral of the following type,

$$I(f_1, \cdots, f_k)(x) = \int_{\mathbb{R}^{nk}} \frac{f_1(l_1) \cdots f_k(l_k) dy_1 \cdots dy_k}{(|y_1| + \cdots + |y_k|)^\lambda},$$

where

$$l_i := l_i(y_1, \cdots, y_k, x) = \sum_{j=1}^{k} A_{i,j} y_j + A_{i,k+1} x$$

are linear combinations of $y_1, \cdots, y_k, x \in \mathbb{R}^n$, $A_{i,j}$ are $n \times n$ matrices such that

1. For each $1 \leq i \leq k$, $A_{i,k+1}$ is invertible,
2. The $nk \times nk$ matrix $(A_{i,j})_{1 \leq i,j \leq k}$ is invertible.

They showed that when $1 < p_i \leq \infty$, $0 < q < \infty$, $0 < \lambda < kn$ and

$$\frac{1}{p_1} + \cdots + \frac{1}{p_k} = \frac{1}{q} + \frac{kn - \lambda}{n},$$

$I(f_1, \cdots, f_k)$ is bounded from $L^{p_1} \times \cdots \times L^{p_k}$ to L^q.

In this paper, we focus on the bilinear case. Denote

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$