

Bilinear Fractional Integral Operators

Ting Chen* and Wenchang Sun

School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China

Received 18 May 2022; Accepted (in revised version) 21 March 2025

Dedicated to the memory of Prof. Donggao Deng on the occasion of his 90th birthday

Abstract. We study the bilinear fractional integral considered by Kenig and Stein, where linear combinations of variables with matrix coefficients are involved. Under more general settings, we give a complete characterization of the corresponding parameters for which the bilinear fractional integral is bounded from $L^{p_1}(\mathbb{R}^{n_1}) \times L^{p_2}(\mathbb{R}^{n_2})$ to $L^q(\mathbb{R}^m)$.

Key Words: Bilinear fractional integrals, Riesz potentials.

AMS Subject Classifications: 42B20

1 Introduction and the main result

The multilinear singular integral operators have been widely studied since Coifman and Meyer's pioneer work [8]. Christ and Journé [7], Grafakos and Torres [15] developed the theory of multilinear Calderón-Zygmund operators. Lacey and Thiele [20,21] proved the boundedness of the bilinear Hilbert transform.

In this paper, we focus on the bilinear fractional integral studied by Kenig and Stein [17], Grafakos and Kalton [10, 13], Grafakos and Lynch [14]. Recall that for $0 < \lambda < 2n$, $f_1 \in L^{p_1}(\mathbb{R}^n)$ and $f_2 \in L^{p_2}(\mathbb{R}^n)$, the bilinear fractional integral of (f_1, f_2) is defined by

$$\int_{\mathbb{R}^{2n}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|x - y_1| + |x - y_2|)^\lambda}.$$

It was shown in [17, Lemma 7] that the bilinear fractional integral is bounded from $L^{p_1} \times L^{p_2}$ to L^q when the indices satisfy certain conditions. We refer to [1-4, 9, 16, 19, 22-24] for some recent advances on the study of bilinear fractional integrals and their applications in PDE. See also [5,6] for a generalization to mixed-norm Lebesgue spaces.

Komori-Furuya [18] gave a complete characterization of the indices for which the multilinear fractional integral is bounded. Here we cite a bilinear version.

*Corresponding author. *Email addresses:* t.chen@nankai.edu.cn (T. Chen), sunwch@nankai.edu.cn (W. Sun)

Proposition 1.1 ([18]). *Suppose that $1 \leq p_1, p_2 \leq \infty$, $0 < \lambda < 2n$ and $1/p_1 + 1/p_2 = 1/q + (2n - \lambda)/n$. Then the norm estimate*

$$\left\| \int_{\mathbb{R}^{2n}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|x - y_1| + |x - y_2|)^\lambda} \right\|_{L_x^q} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}$$

is true if and only if

- (a) either $1 < p_1 < \infty$ or $1 < p_2 < \infty$,
- (b) the index q satisfies

$$\begin{cases} \max\{p_1, p_2\} \leq q < \infty, & \text{if } \min\{p_1, p_2\} = 1, \\ \min\{p_1, p_2\} < q < \infty, & \text{if } \max\{p_1, p_2\} = \infty, \\ 0 < 1/q < 1/p_1 + 1/p_2, & \text{if } 1 < p_1, p_2 < \infty \text{ and } 1/p_1 + 1/p_2 < 1, \\ 0 \leq 1/q < 1/p_1 + 1/p_2, & \text{if } 1 < p_1, p_2 < \infty \text{ and } 1/p_1 + 1/p_2 \geq 1. \end{cases}$$

Kenig and Stein [17, Remark 10] studied the multi-linear fractional integral of the following type,

$$I(f_1, \dots, f_k)(x) = \int_{\mathbb{R}^{nk}} \frac{f_1(l_1) \cdots f_k(l_k) dy_1 \cdots dy_k}{(|y_1| + \cdots + |y_k|)^\lambda},$$

where

$$l_i := l_i(y_1, \dots, y_k, x) = \sum_{j=1}^k A_{i,j}y_j + A_{i,k+1}x$$

are linear combinations of $y_1, \dots, y_k, x \in \mathbb{R}^n$, $A_{i,j}$ are $n \times n$ matrices such that

1. For each $1 \leq i \leq k$, $A_{i,k+1}$ is invertible,
2. The $nk \times nk$ matrix $(A_{i,j})_{1 \leq i,j \leq k}$ is invertible.

They showed that when $1 < p_i \leq \infty$, $0 < q < \infty$, $0 < \lambda < kn$ and

$$\frac{1}{p_1} + \cdots + \frac{1}{p_k} = \frac{1}{q} + \frac{kn - \lambda}{n},$$

$I(f_1, \dots, f_k)$ is bounded from $L^{p_1} \times \cdots \times L^{p_k}$ to L^q .

In this paper, we focus on the bilinear case. Denote

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

As in [17], we suppose that A is invertible. By a change of variable of the form $y \rightarrow A^{-1}y$, we get

$$\begin{aligned} I(f_1, f_2)(x) &= \int_{\mathbb{R}^{2n}} \frac{f_1(A_{11}y_1 + A_{12}y_2 + A_{13}x)f_2(A_{21}y_1 + A_{22}y_2 + A_{23}x)dy_1dy_2}{(|y_1| + |y_2|)^\lambda} \\ &\approx \int_{\mathbb{R}^{2n}} \frac{f_1(y_1 + A_{13}x)f_2(y_2 + A_{23}x)dy_1dy_2}{(|y_1| + |y_2|)^\lambda} \\ &= \int_{\mathbb{R}^{2n}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|A_{13}x - y_1| + |A_{23}x - y_2|)^\lambda}, \end{aligned}$$

where the notation $X \approx Y$ means that there exist positive constants C_1, C_2 such that $C_1Y \leq X \leq C_2Y$. It follows from [17, Remark 10] that $I(f_1, f_2)$ is bounded from $L^{p_1} \times L^{p_2}$ to L^q when both A_{13} and A_{23} are invertible and the indices satisfy certain conditions.

We show that for $I(f_1, f_2)$ to be bounded, A_{13} or A_{23} might be singular. Moreover, we study the bilinear fractional integral in more general settings.

Let n_1, n_2 and m be positive integers, D_i be $n_i \times m$ matrix, $i = 1, 2, 0 < p_1, p_2, q \leq \infty$ and $0 < \lambda < n_1 + n_2$. For $f_i \in L^{p_i}(\mathbb{R}^{n_i}), i = 1, 2$ and $x \in \mathbb{R}^m$, consider the bilinear fractional integral

$$I_{\lambda, D}(f_1, f_2)(x) = \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|D_1x - y_1| + |D_2x - y_2|)^\lambda}. \tag{1.1}$$

We give necessary and sufficient conditions on the matrices D_1 and D_2 and the indices p_1, p_2, q and λ such that $I_{\lambda, D}$ is bounded from $L^{p_1}(\mathbb{R}^{n_1}) \times L^{p_2}(\mathbb{R}^{n_2})$ to $L^q(\mathbb{R}^m)$. Our result generalizes Proposition 1.1.

Theorem 1.1. *Suppose that n_1, n_2 and m are positive integers, that $0 < \lambda < n_1 + n_2$, that $0 < p_1, p_2, q \leq \infty$, and that D_i are $n_i \times m$ matrices, $i = 1, 2$. Denote $r_i = \text{rank}(D_i)$. Then $I_{\lambda, D}$ is bounded from $L^{p_1}(\mathbb{R}^{n_1}) \times L^{p_2}(\mathbb{R}^{n_2})$ to $L^q(\mathbb{R}^m)$ if and only if*

- (a) *The rank of the $(n_1 + n_2) \times m$ matrix $\begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$ is m ;*
- (b) *The indices meet the homogeneity condition $\lambda = n_1/p'_1 + n_2/p'_2 + m/q$;*
- (c) *The index vector $\vec{p} = (p_1, p_2)$ satisfies $p_1, p_2 \geq 1, \#\{i : 1 < p_i < \infty\} \geq 1, p_1 < \infty$ when $r_2 < m$, and $p_2 < \infty$ when $r_1 < m$;*
- (d) *The index q is finite when $\#\{i : 1 < p_i < \infty\} = 1$ or $1/p_1 + 1/p_2 < 1$. Moreover, q satisfies one of the following conditions,*
 - (i) *when $r_1 = r_2 = m$,*

$$\frac{1}{q} \leq \sum_{i: p_i > 1} \frac{1}{p_i},$$

where the equality is accessible only if (1) $\min\{p_1, p_2\} = 1$, or (2) $1 < p_1, p_2 < \infty, n_1, n_2 > m$ and $1/p_1 + 1/p_2 \geq 1$,

(ii) when $r_1 = 0$ and $r_2 = m$ (the case $r_1 = m$ and $r_2 = 0$ is similar),

$$\begin{cases} q \geq p_1, & \text{if } p_2 = 1, \\ q \geq p_2, & \text{if } 1 < p_2 < \infty, \text{ where the equality is accessible only if} \\ & n_2 > m \text{ and } 1 < p_1 \leq p'_2, \end{cases}$$

(iii) when $0 < r_1 < r_2 = m$ or $0 < r_2 < r_1 = m$,

$$\begin{cases} q \geq \max\{p_1, p_2\}, & \text{if } \min\{p_1, p_2\} = 1, \\ q > \min\{p_1, p_2\}, & \text{if } (p_1, r_2) = (\infty, m) \text{ or } (p_2, r_1) = (\infty, m), \\ q \geq a_{p_1, p_2}, & \text{if } 1 < p_1, p_2 < \infty, \end{cases}$$

where $a_{p_1, p_2} = p_1$ when $r_1 = m$, and $a_{p_1, p_2} = p_2$ when $r_2 = m$,

(iv) when $0 < r_1, r_2 < m$,

$$q \geq \max\{p_1, p_2\},$$

where the equality is accessible only if one of the following conditions is satisfied,

- (1) $p_i = 1$ for some i with $r_1 + r_2 > m$ or $r_{3-i} < n_{3-i}$,
- (2) $1 < p_1 \neq p_2 < \infty$,
- (3) $p_1 = p_2$ and $r_1 + r_2 > m$,
- (4) $1 < p_1 = p_2 \leq 2, r_1 + r_2 = m, n_1 > r_1$ and $n_2 > r_2$.

The paper is organized as follows. In Section 2, we give a generalized version of the (linear) fractional integral. In Sections 3 and 4, we give the proof of the necessity and sufficiency of Theorem 1.1, respectively.

2 Preliminary results

In this section, we collect some preliminary results which are used in the proof of Theorem 1.1.

2.1 Operators that commute with translations

It is well known that if a linear operator commutes with translations and is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some $0 < p, q < \infty$, then $p \leq q$. For our purpose, we need a slightly general version, which can be proved similarly to [11, Theorem 2.5.6].

Proposition 2.1. *Suppose that T is a non-zero linear operator which is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^m)$ for some $0 < p, q < \infty$. For $a \in \mathbb{R}$ and $1 \leq i \leq n$, define $f_a(y) = f(y_1, \dots, y_{i-1}, y_i - a, y_{i+1}, \dots, y_n)$. Suppose that for some $1 \leq j \leq m$,*

$$T(f_a)(x) = (Tf)(x_1, \dots, x_{j-1}, x_j - a, x_{j+1}, \dots, x_m).$$

Then we have $p \leq q$.

Proof. For any $f \in L^p$ and $a \in \mathbb{R}$, we have

$$\begin{aligned} & \lim_{a \rightarrow \infty} \|Tf_a + Tf\|_{L^q} \\ &= \lim_{a \rightarrow \infty} \|(Tf)(x_1, \dots, x_{j-1}, x_j - a, x_{j+1}, \dots, x_n) - (Tf)(x)\|_{L^q} \\ &= 2^{1/q} \|Tf\|_{L^q}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \lim_{a \rightarrow \infty} \|Tf_a + Tf\|_{L^q} &\leq \lim_{a \rightarrow \infty} \|T\| \cdot \|f_a + f\|_{L^p} \\ &= 2^{1/p} \|T\| \cdot \|f\|_{L^p}. \end{aligned}$$

Hence

$$2^{1/q} \|Tf\|_{L^q} \leq 2^{1/p} \|T\| \cdot \|f\|_{L^p}.$$

Therefore, $1/q \leq 1/p$. That is, $q \geq p$. □

2.2 Generalized fractional integrals

For $f \in L^p(\mathbb{R}^n)$ and $0 < \lambda < n$, the fractional integral, also known as the Riesz potential, of f is defined by

$$I_\lambda f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x - y|^\lambda}.$$

It was shown that I_λ is bounded from L^p to L^q whenever $1 < p < q < \infty$. We refer to [12, Theorem 1.2.3] or [25, Chapter 5.1] for a proof. In this paper, we consider a generalized version.

Theorem 2.1. *Let n, m be positive integers, $0 < \lambda < n$, $0 < p, q \leq \infty$ and D be an $n \times m$ matrix. Then the norm estimate*

$$\left\| \int_{\mathbb{R}^n} \frac{f(y)dy}{|Dx - y|^\lambda} \right\|_{L_x^q} \lesssim \|f\|_{L^p} \quad (2.1)$$

is true if and only if $\text{rank}(D) = m$, $1 < p < q < \infty$ and $\lambda = n/p' + m/q$.

Proof. Sufficiency. Since $\text{rank}(D) = m$, we have $n \geq m$. There are two cases.

(i) $n = m$.

In this case, D is invertible. The conclusion follows by the boundedness of the Riesz potential.

(ii) $n > m$.

In this case, there exist an $n \times n$ invertible matrix P and an $m \times m$ invertible matrix Q such that $D = P \begin{pmatrix} Q \\ 0 \end{pmatrix}$. Since $|Dx - y| \approx |P^{-1}Dx - P^{-1}y|$, By replacing f with $f(P^{-1}\cdot)$ in (2.1), we get that (2.1) is equivalent to

$$\left\| \int_{\mathbb{R}^n} \frac{f(y)dy}{|P^{-1}Dx - y|^\lambda} \right\|_{L_x^q} \lesssim \|f\|_{L^p}.$$

Denote $y = (y_1, y_2)$, where $y_1 \in \mathbb{R}^m$ and $y_2 \in \mathbb{R}^{n-m}$. By a change of variable of the form $x \rightarrow Q^{-1}x$, we see that the above inequality is equivalent to

$$\left\| \int_{\mathbb{R}^n} \frac{f(y_1, y_2)dy_1dy_2}{(|x - y_1| + |y_2|)^\lambda} \right\|_{L_x^q} \lesssim \|f\|_{L^p}.$$

By Hölder’s inequality, we have

$$\int_{\mathbb{R}^m} \frac{f(y_1, y_2)dy_1dy_2}{(|x - y_1| + |y_2|)^\lambda} \lesssim \int_{\mathbb{R}^m} \frac{\|f(y_1, \cdot)\|_{L_{y_2}^p} dy_1}{|x - y_1|^{\lambda - (n-m)/p'}}.$$

Now we get the conclusion as desired by the boundedness of the Riesz potential.

Necessity. First, we show that $\text{rank}(D) = m$. Assume on the contrary that $\text{rank}(D) < m$. Then there is some invertible $m \times m$ matrix Q such that the last column of DQ is zero. By a change of variable of the form $x \rightarrow Qx$, we get

$$\begin{aligned} \left\| \int_{\mathbb{R}^n} \frac{f(y)dy}{|Dx - y|^\lambda} \right\|_{L_x^q} &= \left\| \int_{\mathbb{R}^n} \frac{f(y)dy}{|DQx - y|^\lambda} \right\|_{L_x^q} \\ &= \left\| \text{a function of } (x^{(1)}, \dots, x^{(m-1)}) \right\|_{L_x^q} = \infty, \end{aligned}$$

which is a contradiction.

By replacing f with $f(\cdot/a)$, we get the homogeneity condition $\lambda = n/p' + m/q$. It remains to show that $1 < p < q < \infty$.

As in the sufficiency part, we have

$$\left\| \int_{\mathbb{R}^n} \frac{f(y_1, y_2)dy_1dy_2}{(|x - y_1| + |y_2|)^\lambda} \right\|_{L_x^q} \lesssim \|f\|_{L^p}, \tag{2.2}$$

where $y_1 \in \mathbb{R}^m$ and $y_2 \in \mathbb{R}^{n-m}$. If $n = m$, then we delete the variable y_2 .

First, we show that $q < \infty$. If $q = \infty$, then we see from (2.2) that for any $h \in L^1$,

$$\int_{\mathbb{R}^{n+m}} \frac{|f(y_1, y_2)h(x)|dx dy_1 dy_2}{(|x - y_1| + |y_2|)^\lambda} \lesssim \|f\|_{L^p} \|h\|_{L^1}.$$

Set $h = (1/\delta^m)\chi_{\{|x| \leq \delta\}}$ and let $\delta \rightarrow 0$. We see from Fatou’s lemma that

$$\int_{\mathbb{R}^n} \frac{|f(y)|dy}{|y|^\lambda} \lesssim \|f\|_{L^p}, \tag{2.3}$$

which is impossible since $1/|y|^\lambda \notin L^{p'}$ when $p \geq 1$ and $(L^p)^* = \{0\}$ when $0 < p < 1$. Hence $q < \infty$.

Next we show that $1 < p < \infty$. If $p \leq 1$, then $\lambda = n/p' + m/q \leq m/q$. By setting $f = \chi_{\{|y_1|+|y_2|\leq 1\}}$ in (2.2), we get

$$\left\| \frac{1}{(1+|x|)^\lambda} \right\|_{L_x^q} < \infty,$$

which is impossible since $\lambda q \leq m$.

If $p = \infty$, by setting $f \equiv 1$, we get

$$\int_{\mathbb{R}^n} \frac{f(y_1, y_2) dy_1 dy_2}{(|x - y_1| + |y_2|)^\lambda} = \infty.$$

Hence $1 < p < \infty$.

Finally, we show that $q > p$. Denote

$$Tf(x) = \int_{\mathbb{R}^n} \frac{f(y_1, y_2) dy_1 dy_2}{(|x - y_1| + |y_2|)^\lambda}.$$

For $z = (0, \dots, 0, a) \in \mathbb{R}^m$, we have

$$\begin{aligned} (Tf(\cdot - z, \cdot))(x) &= \int_{\mathbb{R}^n} \frac{f(y_1 - z, y_2) dy_1 dy_2}{(|x - y_1| + |y_2|)^\lambda} \\ &= (Tf)(x - z). \end{aligned}$$

By Proposition 2.1, we have $p \leq q$.

If $p = q$, then $\lambda = (n - m)/p' + m$. Since $\lambda < n$, we have $m < n$. It follows from (2.2) that for any $h \in L^{q'}$,

$$\left\| \int_{\mathbb{R}^m} \frac{h(x) dx}{(|x - y_1| + |y_2|)^\lambda} \right\|_{L_y^{p'}} \lesssim \|h\|_{L^{q'}}. \tag{2.4}$$

Set $h = \chi_{\{|x|\leq 1\}}$. For $\delta > 0$ small enough and $|y_1|, |y_2| \leq \delta$, we have

$$\begin{aligned} \int_{\mathbb{R}^m} \frac{h(x) dx}{(|x - y_1| + |y_2|)^\lambda} &\geq \int_{|x - y_1| \leq |y_2|} \frac{h(x) dx}{(|x - y_1| + |y_2|)^\lambda} \\ &\gtrsim \frac{1}{|y_2|^{\lambda - m}}. \end{aligned}$$

Since $\lambda - m = (n - m)/p'$, we have

$$\left\| \int_{\mathbb{R}^m} \frac{h(x) dx}{(|x - y_1| + |y_2|)^\lambda} \right\|_{L_y^{p'}} = \infty,$$

which is a contradiction. This completes the proof. □

Next we show that when $|x - y|$ is replaced by $|x| + |y|$ in the definition of I_λ , then even for $p = q$, the operator is bounded from L^p to L^q .

Lemma 2.1. *Suppose that $0 < p, q \leq \infty$ and $\lambda > 0$. Then the inequality*

$$\left\| \int_{\mathbb{R}^n} \frac{f(y)dy}{(|x| + |y|)^\lambda} \right\|_{L^q(\mathbb{R}^m)} \lesssim \|f\|_{L^p}, \quad f \in L^p(\mathbb{R}^n), \tag{2.5}$$

is true if and only if $\lambda = n/p' + m/q$ and $1 < p \leq q < \infty$.

As a result, for $1 \leq p_1, p_2 \leq \infty$ with $(p_1, p_2) \neq (1, 1)$,

$$\left| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|y_1| + |y_2|)^{n_1/p'_1+n_2/p'_2}} \right| \lesssim \|f_1\|_{L^{p_1}(\mathbb{R}^{n_1})} \|f_2\|_{L^{p_2}(\mathbb{R}^{n_2})} \tag{2.6}$$

if and only if $1 < p_1, p_2 < \infty$ and $1/p_1 + 1/p_2 \geq 1$.

Proof. Denote

$$T_\lambda f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{(|x| + |y|)^\lambda}.$$

Necessity. As in the proof of Theorem 2.1, by replacing f with $f(\cdot/a)$, we get the homogeneity condition $\lambda = n/p' + m/q$. It remains to show that $1 < p \leq q < \infty$.

If $q = \infty$, then for any $f \in L^p$ and $h \in L^1(\mathbb{R}^m)$,

$$\int_{\mathbb{R}^{m+n}} \frac{|f(y)h(x)|dx dy}{(|x| + |y|)^\lambda} \lesssim \|f\|_{L^p} \|h\|_{L^1}.$$

By setting $h = (1/\delta^m)\chi_{\{|x| \leq \delta\}}$ and letting $\delta \rightarrow 0$, we get (2.3), which is impossible. Hence $q < \infty$.

If $p \leq 1$, then $\lambda = n/p' + m/q \leq m/q$. Set $f = \chi_{\{|y| \leq 1\}}$. We have

$$|T_\lambda f(x)| \geq \frac{1}{(1 + |x|)^\lambda}.$$

Hence $T_\lambda f \notin L^q$, which is a contradiction.

It remains to show that $p \leq q$. Set

$$f(y) = \frac{\chi_{\{|y| \leq 1/2\}}(y)}{|y|^{n/p}(\log 1/|y|)^{(1+\varepsilon)/p'}}$$

where $\varepsilon > 0$ is a constant. We have $f \in L^p$. For $\delta > 0$ small enough and $|x| \leq \delta$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{f(y)dy}{(|x| + |y|)^\lambda} &\geq \int_{|x|^2 \leq |y| \leq |x|} \frac{dy}{(|x| + |y|)^{n/p'+m/q} |y|^{n/p} (\log 1/|y|)^{(1+\varepsilon)/p'}} \\ &\gtrsim \frac{1}{|x|^{m/q} (\log 1/|x|)^{(1+\varepsilon)/p'}.} \end{aligned}$$

If $q < p$, we can choose $\varepsilon > 0$ small enough such that $q(1 + \varepsilon)/p < 1$. Consequently,

$$\left\| \int_{\mathbb{R}^n} \frac{f(y)dy}{(|x| + |y|)^\lambda} \right\|_{L_x^q} = \infty,$$

which contradicts the assumption. Hence $p \leq q$.

Sufficiency. Define the operator S by

$$Sf(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{(|x| + |y|)^\lambda}, \quad x \in \mathbb{R}^m.$$

Let $1 < r, s < \infty$ be such that $n/r' + m/s = \lambda$. For any $f \in L^r$, it follows from Hölder's inequality that

$$|Sf(x)| \leq \frac{\|f\|_{L^r}}{|x|^{m/s}}.$$

Hence

$$\|Sf\|_{L^{s,\infty}} \lesssim \|f\|_{L^r}.$$

By the interpolation theorem, S is bounded from L^p to L^q whenever $n/p' + m/q = \lambda$ and $q \geq p$. Hence (2.5) is true. This completes the proof. \square

3 Proof of Theorem 1.1: the necessity

In this section, we give a proof of the necessity part in Theorem 1.1. We prove the conclusion in several steps.

(S1) We show that $\text{rank}_{\binom{D_1}{D_2}} = m$.

Since D_i are $n_i \times m$ matrices, $i = 1, 2$, we have $\text{rank}_{\binom{D_1}{D_2}} \leq m$. Assume that $\text{rank}_{\binom{D_1}{D_2}} < m$. Then there is some $m \times m$ invertible matrix P such that the last column of $\binom{D_1}{D_2}P$ is zero. By a change of variable of the form $x \rightarrow Px$, we get

$$\begin{aligned} \|I_{\lambda,D}(f_1, f_2)\|_{L^q} &= \left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|D_1x - y_1| + |D_2x - y_2|)^\lambda} \right\|_{L^q} \\ &\approx \left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|D_1Px - y_1| + |D_2Px - y_2|)^\lambda} \right\|_{L^q} \\ &= \left\| \text{a function of } (x^{(1)}, \dots, x^{(m-1)}) \right\|_{L^q} = \infty, \end{aligned}$$

which is a contradiction.

(S2) We show that the indices p_1, p_2, q and λ satisfy the homogeneity condition.

For $a > 0$, set $f_{i,a} = f_i(\cdot/a)$, $i = 1, 2$. We have

$$\begin{aligned} I_{\lambda,D}(f_{1,a}, f_{2,a})(x) &= \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1/a)f_2(y_2/a)dy_1dy_2}{(|D_1x - y_1| + |D_2x - y_2|)^\lambda} \\ &= \int_{\mathbb{R}^{n_1+n_2}} \frac{a^{n_1+n_2}f_1(y_1)f_2(y_2)dy_1dy_2}{(|D_1x - ay_1| + |D_2x - ay_2|)^\lambda} \\ &= \int_{\mathbb{R}^{n_1+n_2}} \frac{a^{n_1+n_2-\lambda}f_1(y_1)f_2(y_2)dy_1dy_2}{(|D_1x/a - y_1| + |D_2x/a - y_2|)^\lambda} \\ &= a^{n_1+n_2-\lambda}I_{\lambda,D}(f_1, f_2)(x/a). \end{aligned}$$

Since $I_{\lambda,D}$ is bounded, we have

$$\|a^{n_1+n_2-\lambda}I_{\lambda,D}(f_1, f_2)(x/a)\|_{L^q} \lesssim C\|f_{1,a}\|_{L^{p_1}}\|f_{2,a}\|_{L^{p_2}},$$

where $C = \|I_{\lambda,D}\|_{L^{p_1} \times L^{p_2} \rightarrow L^q}$. Hence

$$a^{n_1+n_2-\lambda+m/q}\|I_{\lambda,D}(f_1, f_2)\|_{L^q} \lesssim a^{n_1/p_1+n_2/p_2}C\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.$$

Since $a > 0$ is arbitrary, we have $n_1 + n_2 - \lambda + m/q = n_1/p_1 + n_2/p_2$. Thus $\lambda = n_1/p'_1 + n_2/p'_2 + m/q$.

(S3) We show that $p_i \geq 1$, $i = 1, 2$.

Without loss of generality, assume that $p_1 < 1$. Then $\lambda < n_2 - n_2/p_2 + m/q$. Hence there is some $\alpha > n_2/p_2$ such that $\lambda + \alpha - n_2 < m/q$. Set

$$f_1(y_1) = \chi_{\{|y_1| \leq 1\}}(y_1) \quad \text{and} \quad f_2(y_2) = \frac{1}{(1 + |y_2|)^\alpha}.$$

Then we have $f_i \in L^{p_i}$. However,

$$\begin{aligned} I_{\lambda,D}(f_1, f_2)(x) &= \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|D_1x - y_1| + |D_2x - y_2|)^\lambda} \\ &\gtrsim \int_{\mathbb{R}^{n_2}} \frac{f_2(y_2)dy_2}{(1 + |x| + |y_2|)^\lambda} \\ &\geq \int_{\mathbb{R}^{n_2}} \frac{dy_2}{(1 + |x| + |y_2|)^{\lambda+\alpha}} \\ &\approx \frac{1}{(1 + |x|)^{\lambda+\alpha-n_2}}, \end{aligned}$$

which is not in L^q when $q < \infty$.

When $q = \infty$ and $p_2 < \infty$, the above choices of f_1 and f_2 lead to

$$I_{\lambda,D}(f_1, f_2)(x) \gtrsim \int_{\mathbb{R}^{n_2}} \frac{dy_2}{(1 + |x| + |y_2|)^{\lambda+\alpha}} = \infty.$$

When $p_2 = q = \infty$, we have $\lambda < n_2$. Set $f_1 = \chi_{\{|y_1| \leq 1\}}$ and $f_2 \equiv 1$. We get

$$\begin{aligned} I_{\lambda,D}(f_1, f_2)(x) &= \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|D_1x - y_1| + |D_2x - y_2|)^\lambda} \\ &\gtrsim \int_{\mathbb{R}^{n_2}} \frac{dy_2}{(1 + |x| + |y_2|)^\lambda} = \infty. \end{aligned}$$

Again, we get a contradiction. Hence $p_1, p_2 \geq 1$.

(S4) We show that either $1 < p_1 < \infty$ or $1 < p_2 < \infty$.

Assume on the contrary that $p_i = 1$ or the infinity. There are three cases.

(i) $p_1 = p_2 = 1$. In this case, $\lambda = m/q$. Hence $q < \infty$. Set $f_i = \chi_{\{|y_i| \leq 1\}}$, $i = 1, 2$. We have

$$\begin{aligned} I_{\lambda,D}(f_1, f_2)(x) &= \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|D_1x - y_1| + |D_2x - y_2|)^\lambda} \\ &\gtrsim \frac{1}{(1 + |x|)^\lambda}. \end{aligned}$$

Therefore, $I_{\lambda,D}(f_1, f_2) \notin L^q$, which is a contradiction.

(ii) $p_1 = p_2 = \infty$. In this case, $\lambda = n_1/p'_1 + n_2/p'_2 + m/q \geq n_1 + n_2$, which contradicts the hypothesis.

(iii) One of p_1 and p_2 equals 1, and the other equals ∞ . Without loss of generality, assume that $p_1 = 1$ and $p_2 = \infty$. Then we have $\lambda = n_2 + m/q$. Set $f_2 \equiv 1$.

If $q = \infty$, then

$$I_{\lambda,D}(f_1, f_2)(x) = \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)dy_1dy_2}{(|D_1x - y_1| + |D_2x - y_2|)^{n_2}} = \infty.$$

If $q < \infty$, we have

$$\begin{aligned} I_{\lambda,D}(f_1, f_2)(x) &= \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)dy_1dy_2}{(|D_1x - y_1| + |D_2x - y_2|)^\lambda} \\ &\approx \int_{\mathbb{R}^{n_1}} \frac{f_1(y_1)dy_1}{|D_1x - y_1|^{m/q}}. \end{aligned}$$

Hence,

$$\left\| \int_{\mathbb{R}^{n_1}} \frac{f_1(y_1)dy_1}{|D_1x - y_1|^{m/q}} \right\|_{L^q} \lesssim \|f_1\|_1,$$

which is impossible, thanks to Theorem 2.1.

In both cases, we get contradictions. Hence there is some i such that $1 < p_i < \infty$.

(S5) We show that $p_1 < \infty$ when $r_2 < m$, and $p_2 < \infty$ when $r_1 < m$.

Assume that $r_2 < m$ and $p_1 = \infty$. Set $f_1 \equiv 1$. We see from

$$\|I_{\lambda,D}(f_1, f_2)\|_{L^q} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}$$

that

$$\left\| \int_{\mathbb{R}^{n_2}} \frac{f_2(y_2) dy_2}{|Dx - y_2|^{n_2/p'_2 + m/q}} \right\|_{L^q_x} \lesssim \|f_2\|_{L^{p_2}},$$

which contradicts Theorem 2.1 since $r_2 = \text{rank}(D_2) < m$.

Similarly we get that $p_2 < \infty$ when $r_1 < m$.

(S6) We show that $q < \infty$ when one of the three conditions is satisfied, $\min\{p_1, p_2\} = 1$, $\max\{p_1, p_2\} = \infty$ or $1/p_1 + 1/p_2 < 1$.

Assume on the contrary that $q = \infty$. Then $\lambda = n_1/p'_1 + n_2/p'_2$. For any $h \in L^1$,

$$\int_{\mathbb{R}^{n_1+n_2+m}} \frac{|f_1(y_1)f_2(y_2)h(x)| dx dy_1 dy_2}{(|D_1x - y_1| + |D_2x - y_2|)^\lambda} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \|h\|_{L^1}.$$

By setting $h = (1/\delta^m)\chi_{\{|x| \leq \delta\}}$ and letting $\delta \rightarrow 0$, we get

$$\int_{\mathbb{R}^{n_1+n_2}} \frac{|f_1(y_1)f_2(y_2)| dy_1 dy_2}{(|y_1| + |y_2|)^{n_1/p'_1 + n_2/p'_2}} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}.$$

By Lemma 2.1, $1 < p_1, p_2 < \infty$ and $1/p_1 + 1/p_2 \geq 1$, which contradicts the assumption. Hence $q < \infty$.

In the following subsections, we prove the rest part in several cases.

3.1 The case $r_1 = r_2 = m$

Since $\text{rank}(D_i) = m$ and D_i are $n_i \times m$ matrices, we have $n_1, n_2 \geq m$. There are $n_i \times n_i$ invertible matrices $P_i, i = 1, 2$, such that

$$P_1 D_1 = \begin{pmatrix} I_m \\ 0_{(n_1-m) \times m} \end{pmatrix} \quad \text{and} \quad P_2 D_2 = \begin{pmatrix} I_m \\ 0_{(n_2-m) \times m} \end{pmatrix},$$

where I_m stands for the $m \times m$ identity matrix. Note that

$$|D_1x - y_1| + |D_2x - y_2| \approx |P_1 D_1x - P_1 y_1| + |P_2 D_2x - P_2 y_2|.$$

We have

$$\begin{aligned} & \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2) dy_1 dy_2}{(|D_1x - y_1| + |D_2x - y_2|)^\lambda} \\ & \approx \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2) dy_1 dy_2}{(|P_1 D_1x - P_1 y_1| + |P_2 D_2x - P_2 y_2|)^\lambda}. \end{aligned}$$

By replacing $f_1(P_1 \cdot)$ and $f_2(P_2 \cdot)$ for f_1 and f_2 , respectively, we get that the boundedness of $I_{\lambda,D}$ is equivalent to

$$\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|P_1D_1x - y_1| + |P_2D_2x - y_2|)^\lambda} \right\|_{L^q} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}.$$

Denote $y_i = (y_{i1}, y_{i2})$, where $y_{i1} \in \mathbb{R}^m$ and $y_{i2} \in \mathbb{R}^{n_i-m}$, $i = 1, 2$. The above inequality becomes

$$\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_{11}, y_{12})f_2(y_{21}, y_{22})dy_1dy_2}{(|x - y_{11}| + |x - y_{21}| + |y_{12}| + |y_{22}|)^\lambda} \right\|_{L^q} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}. \tag{3.1}$$

In the followings we prove the conclusion in several steps.

(i) We show that $q \geq \max\{p_1, p_2\}$ when $\min\{p_1, p_2\} = 1$.

Without loss of generality, we assume that $p_1 = 1$. In this case, $1 < p_2 < \infty$ and $\lambda = n_2/p_2' + m/q$.

Setting $f_1 = (1/\delta^{n_1})\chi_{\{|y_1| \leq \delta\}}$ and letting $\delta \rightarrow 0$ in (3.1), we see from Fatou's lemma that

$$\left\| \int_{\mathbb{R}^{n_2}} \frac{f_2(y_2)dy_2}{(|x| + |y_2|)^\lambda} \right\|_{L^q} \lesssim \|f_2\|_{L^{p_2}}, \quad \forall f_2 \in L^{p_2}. \tag{3.2}$$

By Lemma 2.1, $q \geq p_2$.

(ii) We show that $q > \min\{p_1, p_2\}$ when $\max\{p_1, p_2\} = \infty$.

Without loss of generality, assume that $p_1 = \infty$. Then $1 < p_2 < \infty$ and $\lambda = n_1 + n_2/p_2' + m/q$. Setting $f_1 \equiv 1$ in (3.1), we have

$$\left\| \int_{\mathbb{R}^{n_2}} \frac{f_2(y_{21}, y_{22})dy_2}{(|x - y_{21}| + |y_{22}|)^{n_2/p_2' + m/q}} \right\|_{L^q} \lesssim \|f_2\|_{L^{p_2}}, \quad \forall f_2 \in L^{p_2}.$$

Now we see from Theorem 2.1 that $p_2 < q < \infty$. In the followings we assume that $1 < p_1, p_2 < \infty$.

(iii) We prove that $1/q \leq 1/p_1 + 1/p_2$.

Let

$$f_i(y_i) = \frac{\chi_{\{|y_i| \leq 1/2\}}(y_i)}{|y_i|^{n_i/p_i} (\log 1/|y_i|)^{(1+\varepsilon)/p_i}},$$

where $\varepsilon > 0$ is a constant. We have $f_i \in L^{p_i}$, $i = 1, 2$. When $|x|$ is small enough, we have

$$\begin{aligned} & \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_{11}, y_{12})f_2(y_{21}, y_{22})dy_1dy_2}{(|x - y_{11}| + |x - y_{21}| + |y_{12}| + |y_{22}|)^\lambda} \\ & \geq \int_{\substack{|x|^2 \leq |y_i| \leq |x| \\ 1 \leq i \leq 2}} \frac{f_1(y_{11}, y_{12})f_2(y_{21}, y_{22})dy_1dy_2}{(|x - y_{11}| + |x - y_{21}| + |y_{12}| + |y_{22}|)^\lambda} \\ & \gtrsim \frac{1}{|x|^{m/q} (\log 1/|x|)^{(1+\varepsilon)(1/p_1+1/p_2)}}. \end{aligned}$$

If $1/q > 1/p_1 + 1/p_2$, then there is some $\varepsilon > 0$ such that $(1 + \varepsilon)(1/p_1 + 1/p_2)q < 1$. Consequently,

$$\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_{11}, y_{12})f_2(y_{21}, y_{22})dy_1dy_2}{(|x - y_{11}| + |x - y_{21}| + |y_{12}| + |y_{22}|)^\lambda} \right\|_{L^q} = \infty,$$

which contradicts (3.1). Hence $1/q \leq 1/p_1 + 1/p_2$.

(iv) We prove that $1/q < 1/p_1 + 1/p_2$ when $\min\{n_1, n_2\} = m$ or $1/p_1 + 1/p_2 < 1$. Assume on the contrary that $1/q = 1/p_1 + 1/p_2$. There are three cases.

(a) $n_1 = n_2 = m$.

In this case, $\lambda = n_1/p'_1 + n_2/p'_2 + m/q = n_1 + n_2$, which contradicts the hypothesis.

(b) $\min\{n_1, n_2\} = m$ and $\max\{n_1, n_2\} > m$.

Without loss of generality, we assume that $n_1 = m$ and $n_2 > m$. Denote $y_2 = (y_{21}, y_{22})$, where $y_{21} \in \mathbb{R}^m$ and $y_{22} \in \mathbb{R}^{n_2-m}$. Let

$$f_1(y_1) = \chi_{\{|y_1| \leq 1\}}(y_1) \quad \text{and} \quad f_2(y_2) = \frac{\chi_{\{|y_2| \leq 1/2\}}(y_2)}{|y_{22}|^{(n_2-m)/p_2}(\log 1/|y_{22}|)^{(1+\varepsilon)/p_2}},$$

where $\varepsilon > 0$ is a constant such that $(1 + \varepsilon)/p_2 < 1$.

For $\delta > 0$ small enough and $|x| \leq \delta$, we have

$$\begin{aligned} & \int_{\mathbb{R}^{m+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|x - y_1| + |x - y_{21}| + |y_{22}|)^\lambda} \\ & \geq \int_{\substack{|y_1-x| \leq |y_{22}| \\ |y_{21}-x| \leq |y_{22}| \\ |y_{22}| \leq \delta}} \frac{dy_1dy_{21}}{(|x - y_1| + |x - y_{21}| + |y_{22}|)^\lambda} \\ & \quad \times \frac{dy_{22}}{|y_{22}|^{(n_2-m)/p_2}(\log(1/|y_{22}|))^{(1+\varepsilon)/p_2}} \\ & \gtrsim \int_{|y_{22}| \leq \delta} \frac{dy_{22}}{|y_{22}|^{n_2-m}(\log(1/|y_{22}|))^{(1+\varepsilon)/p_2}} = \infty. \end{aligned}$$

Hence $\|I_{\lambda,D}(f_1, f_2)\|_{L^q} = \infty$, which is a contradiction.

(c) $\min\{n_1, n_2\} > m$ and $1/p_1 + 1/p_2 < 1$.

In this case, $q > 1$. we see from (3.1) that for any $h \in L^{q'}$,

$$\left| \int_{\mathbb{R}^{n_1+n_2+m}} \frac{f_1(y_{11}, y_{12})f_2(y_{21}, y_{22})h(x)dx dy_1 dy_2}{(|x - y_{11}| + |x - y_{21}| + |y_{12}| + |y_{22}|)^\lambda} \right| \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \|h\|_{L^{q'}}.$$

Hence

$$\left\| \int_{\mathbb{R}^{m+n_1}} \frac{f_1(y_{11}, y_{12})h(x)dx dy_1}{(|x - y_{11}| + |x - y_{21}| + |y_{12}| + |y_{22}|)^\lambda} \right\|_{L^{p'_2}} \lesssim \|f_1\|_{L^{p_1}} \|h\|_{L^{q'}}. \tag{3.3}$$

Take

$$h(x) = \chi_{\{|x| \leq 1\}}(x) \quad \text{and} \quad f_1(y_1) = \frac{\chi_{\{|y_1| \leq 1/2\}}(y_1)}{|y_{12}|^{(n_1-m)/p_1} (\log 1/|y_{12}|)^{(1+\varepsilon)/p_1}}.$$

For $\delta > 0$ small enough and $|y_2| \leq \delta$, we have

$$\begin{aligned} & \int_{\mathbb{R}^{m+n_1}} \frac{f_1(y_{11}, y_{12})h(x)dx dy_1}{(|x - y_{11}| + |x - y_{21}| + |y_{12}| + |y_{22}|)^\lambda} \\ & \gtrsim \int_{\substack{|x-y_{21}| \leq |y_{22}| \\ |y_{11}-y_{21}| \leq |y_{22}| \\ |y_{22}|^2 \leq |y_{12}| \leq |y_{22}|}} \frac{dx dy_1}{(|x - y_{11}| + |x - y_{21}| + |y_{12}| + |y_{22}|)^\lambda} \\ & \quad \times \frac{1}{|y_{12}|^{(n_1-m)/p_1} (\log(1/|y_{12}|))^{(1+\varepsilon)/p_1}} \\ & \gtrsim \frac{1}{|y_{22}|^{(n_2-m)/p'_2} (\log(1/|y_{22}|))^{(1+\varepsilon)/p_1}}. \end{aligned}$$

Since $1/p_1 < 1/p'_2$, we can choose $\varepsilon > 0$ small enough such that $p'_2(1 + \varepsilon)/p_1 < 1$. Thus

$$\left\| \frac{\chi_{\{|y_2| \leq \delta\}}(y_2)}{|y_{22}|^{(n_2-m)/p'_2} (\log(1/|y_{22}|))^{(1+\varepsilon)/p_1}} \right\|_{L^{p'_2}_{y_2}} = \infty,$$

which contradicts (3.3).

3.2 The case $r_1 = 0$ and $r_2 = m$

In this case, $I_{\lambda,D}$ is bounded if and only if

$$\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1 dy_2}{(|y_1| + |x - y_{21}| + |y_{22}|)^\lambda} \right\|_{L^q_x} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}. \tag{3.4}$$

First, we show that $q \geq p_2$. For $z = (0, \dots, 0, a) \in \mathbb{R}^m$, where only the last coordinate is nonzero, set $f_{2,z}(y_{21}, y_{22}) = f_2(y_{21} - z, y_{22})$. We have

$$I_{\lambda,D}(f_1, f_{2,z})(x) = I_{\lambda,D}(f_1, f_2)(x - z).$$

By Proposition 2.1, $q \geq p_2$.

Now we see from (3.4) that $I_{\lambda,D}$ is bounded if and only if for any $f_1 \in L^{p_1}$, $f_2 \in L^{p_2}$ and $h \in L^{q'}$,

$$\left| \int_{\mathbb{R}^{n_1+n_2+m}} \frac{f_1(y_1)f_2(y_2)h(x)dx dy_1 dy_2}{(|y_1| + |x - y_{21}| + |y_{22}|)^\lambda} \right| \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \|h\|_{L^{q'}}. \tag{3.5}$$

There are three cases.

(i) $p_2 = 1$.

In this case, $1 < p_1 < \infty$. Set $f_2 = (1/\delta^{n_2})\chi_{\{|y_2| \leq \delta\}}$. By letting $\delta \rightarrow 0$, we see from (3.5) that

$$\left| \int_{\mathbb{R}^{n_1+m}} \frac{f_1(y_1)h(x)dx dy_1}{(|y_1| + |x|)^\lambda} \right| \lesssim \|f_1\|_{L^{p_1}} \|h\|_{L^{q'}}.$$

By Lemma 2.1, $q \geq p_1$.

(ii) $p_1 = 1$ or $p_1 = \infty$.

In this case, $1 < p_2 < \infty$. If $p_1 = 1$, by setting $f_1 = (1/\delta^{n_1})\chi_{\{|y_1| \leq \delta\}}$ and letting $\delta \rightarrow 0$, we see from (3.5) that

$$\left| \int_{\mathbb{R}^{n_2+m}} \frac{f_2(y_2)h(x)dx dy_2}{(|x - y_{21}| + |y_{22}|)^{n_2/p_2+m/q}} \right| \lesssim \|f_2\|_{L^{p_2}} \|h\|_{L^{q'}}. \tag{3.6}$$

Now we see from Theorem 2.1 that $q > p_2$.

If $p_1 = \infty$, by setting $f_1 \equiv 1$, we also have (3.6). Hence $q > p_2$.

(iii) $1 < p_1, p_2 < \infty$.

First, we prove that $q > p_2$ when $n_2 = m$. Assume on the contrary that $q = p_2$. Then $\lambda = n_1/p_1' + m$. In this case, (3.4) turns out to be

$$\left\| \int_{\mathbb{R}^{n_1+m}} \frac{f_1(y_1)f_2(y_2)dy_1 dy_2}{(|y_1| + |x - y_2|)^\lambda} \right\|_{L_x^q} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}. \tag{3.7}$$

Set

$$f_1(y_1) = \frac{\chi_{\{|y_1| \leq 1/2\}}(y_1)}{|y_1|^{n_1/p_1}(\log 1/|y_1|)^{(1+\varepsilon)/p_1}} \quad \text{and} \quad f_2 = \chi_{\{|y_2| \leq 1\}},$$

where $\varepsilon > 0$ satisfies $1 + \varepsilon < p_1$. We have $f_1 \in L^{p_1}$. For $\delta > 0$ small enough and $|x|, |y_2| \leq \delta$, we have

$$\begin{aligned} & \int_{\mathbb{R}^{n_1+m}} \frac{f_1(y_1)f_2(y_2)dy_1 dy_2}{(|y_1| + |x - y_2|)^\lambda} \\ & \geq \int_{\substack{|x-y_2|^2 \leq |y_1| \leq |x-y_2| \\ |y_2| \leq \delta}} \frac{dy_1 dy_2}{(|y_1| + |x - y_2|)^{n_1/p_1'+m} |y_1|^{n_1/p_1} (\log 1/|y_1|)^{(1+\varepsilon)/p_1}} \\ & \gtrsim \int_{|y_2| \leq \delta} \frac{dy_2}{|x - y_2|^m (\log 1/|x - y_2|)^{(1+\varepsilon)/p_1}} = \infty, \end{aligned}$$

which contradicts (3.7). Hence $q > p_2$.

Next we prove that $q > p_2$ when $n_2 > m$ and $1/p_1 + 1/p_2 < 1$.

Again, assume that $q = p_2$. Set

$$f_1(y_1) = \frac{\chi_{\{|y_1| \leq 1/2\}}(y_1)}{|y_1|^{n_1/p_1} (\log 1/|y_1|)^{(1+\varepsilon)/p_1}},$$

$$f_2(y_2) = \frac{\chi_{\{|y_2| \leq 1/2\}}(y_2)}{|y_{22}|^{(n_2-m)/p_2} (\log 1/|y_{22}|)^{(1+\varepsilon)/p_2}},$$

where $\varepsilon > 0$ satisfying $(1 + \varepsilon)/p_1 + (1 + \varepsilon)/p_2 < 1$. Then we have $f_i \in L^{p_i}$. Moreover, for δ small enough,

$$\begin{aligned} & \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|y_1| + |x - y_{21}| + |y_{22}|)^\lambda} \\ & \geq \int_{\substack{|y_{22}|^2 \leq |y_1| \leq |y_{22}| \\ |x-y_{21}| \leq |y_{22}| \\ |y_{22}| \leq \delta}} \frac{dy_1dy_{21}}{(|y_1| + |x - y_{21}| + |y_{22}|)^\lambda} \\ & \quad \times \frac{dy_{22}}{|y_1|^{n_1/p_1} (\log 1/|y_1|)^{(1+\varepsilon)/p_1} |y_{22}|^{(n_2-m)/p_2} (\log 1/|y_{22}|)^{(1+\varepsilon)/p_2}} \\ & \gtrsim \int_{|y_{22}| \leq \delta} \frac{dy_{22}}{|y_{22}|^{n_2-m} (\log 1/|y_{22}|)^{(1+\varepsilon)(1/p_1+1/p_2)}} = \infty. \end{aligned}$$

Again, we get a contradiction.

3.3 The case $0 < r_1 < r_2 = m$ or $0 < r_2 < r_1 = m$

We consider only the case $0 < r_1 < r_2 = m$. The other case can be proved similarly.

Since $\text{rank}(D_1) = r_1 < m$, there exist invertible matrices P_1 and Q such that

$$P_1 D_1 Q = \begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix},$$

where I_r is the $r \times r$ identity matrix. Since $\text{rank}(D_2 Q) = \text{rank}(D_2) = m$, there exists an $n_2 \times n_2$ invertible matrix P_2 such that

$$P_2 D_2 Q = \begin{pmatrix} I_m \\ 0 \end{pmatrix},$$

where I_m is the $m \times m$ identity matrix.

Note that

$$|D_1 x - y_1| + |D_2 x - y_2| \approx |P_1 D_1 x - P_1 y_1| + |P_2 D_2 x - P_2 y_2|.$$

By a change of variable of the form $x \rightarrow Qx$ and replacing $f_1(P_1 \cdot)$ and $f_2(P_2 \cdot)$ for f_1 and f_2 respectively, we know that

$$\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|D_1 x - y_1| + |D_2 x - y_2|)^\lambda} \right\|_{L^q_x} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}$$

is equivalent to

$$\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|x_1 - y_{11}| + |y_{12}| + |x - y_{21}| + |y_{22}|)^\lambda} \right\|_{L_x^q} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}, \quad (3.8)$$

where $x = (x_1, x_2)$, $x_1 \in \mathbb{R}^{r_1}$, $x_2 \in \mathbb{R}^{m-r_1}$, $y_{i1} \in \mathbb{R}^{r_i}$ and $y_{i2} \in \mathbb{R}^{n_i-r_i}$. If $n_1 = r_1$ or $n_2 = r_2$, then we delete the variable y_{12} or y_{22} accordingly.

The same arguments as in the case $(r_1, r_2) = (0, m)$ show that $q \geq p_2$. It remains to show that $q \geq p_1$ when $p_2 = 1$, and $q > p_2$ when $p_1 = \infty$.

Note that (3.8) is equivalent to

$$\left| \int_{\mathbb{R}^{n_1+n_2+m}} \frac{f_1(y_1)f_2(y_2)h(x)dx dy_1 dy_2}{(|x_1 - y_{11}| + |y_{12}| + |x - y_{21}| + |y_{22}|)^\lambda} \right| \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \|h\|_{L^{q'}}. \quad (3.9)$$

There are two cases.

(i) $p_2 = 1$.

In this case, $1 < p_1 < \infty$. Set $f_2 = (1/\delta^{n_2})\chi_{\{|y_2| \leq \delta\}}$. By letting $\delta \rightarrow 0$, we see from (3.9) that

$$\left| \int_{\mathbb{R}^{n_1+m}} \frac{f_1(y_1)h(x)dx dy_1}{(|x_1 - y_{11}| + |y_{12}| + |x|)^\lambda} \right| \lesssim \|f_1\|_{L^{p_1}} \|h\|_{L^{q'}}.$$

Note that $|x_1 - y_{11}| + |y_{12}| + |x| \approx |x| + |y_1|$. The above inequality is equivalent to

$$\left| \int_{\mathbb{R}^{n_1+m}} \frac{f_1(y_1)h(x)dx dy_1}{(|y_1| + |x|)^\lambda} \right| \lesssim \|f_1\|_{L^{p_1}} \|h\|_{L^{q'}}, \quad \forall f_1 \in L^{p_1}, \quad h \in L^{q'}.$$

By Lemma 2.1, $p_1 \leq q$.

(ii) $p_1 = \infty$.

In this case, $1 < p_2 < \infty$. by setting $f_1 \equiv 1$, we see from (3.9) that

$$\left| \int_{\mathbb{R}^{n_2+m}} \frac{f_2(y_2)h(x)dx dy_2}{(|x - y_{21}| + |y_{22}|)^{n_2/p_2+m/q}} \right| \lesssim \|f_2\|_{L^{p_2}} \|h\|_{L^{q'}}. \quad (3.10)$$

By Theorem 2.1, we have $p_2 < q$.

3.4 The case $0 < r_1, r_2 < m$

Since $I_{\lambda,D}$ is bounded, we have

$$\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|D_1x - y_1| + |D_2x - y_2|)^\lambda} \right\|_{L_x^q} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}. \quad (3.11)$$

Denote $D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$. Since $r_1 = \text{rank}(D_1) < m$, there is some $m \times m$ invertible matrix Q_1 such that the last $m - r_1$ columns of D_1Q_1 are zero vectors. On the other hand, since

$\text{rank}(D) = m$, the last $m - r_1$ columns of D_2Q_1 must be linearly independent. Hence there exist $1 \leq j_1 < \dots < j_{r_1+r_2-m} \leq r_1$ such that the j_1 -th, \dots , the $j_{r_1+r_2-m}$ -th columns and the last $m - r_1$ columns of D_2Q_1 are linearly independent. Consequently, there is some $m \times m$ invertible matrix Q_2 such that the first $m - r_2$ columns of $D_2Q_1Q_2$ are zero vectors and the last $m - r_1$ columns of $D_1Q_1Q_2$ are zero vectors. That is,

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} Q_1 Q_2 = \begin{pmatrix} * & * & 0_{n_1 \times (m-r_1)} \\ 0_{n_2 \times (m-r_2)} & * & * \end{pmatrix}.$$

Since $\text{rank}(D_iQ_1Q_2) = \text{rank}(D_i)$, there exist $n_1 \times n_1$ invertible matrix P_1 and $n_2 \times n_2$ invertible matrix P_2 such that

$$P_1 D_1 Q_1 Q_2 = \begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_2 D_2 Q_1 Q_2 = \begin{pmatrix} 0 & I_{r_2} \\ 0 & 0 \end{pmatrix}.$$

Note that

$$|D_1x - y_1| + |D_2x - y_2| \approx |P_1D_1x - P_1y_1| + |P_2D_2x - P_2y_2|.$$

By replacing $f_1(P_1 \cdot)$ and $f_2(P_2 \cdot)$ for f_1 and f_2 respectively and a change of variables of the form $(x, y_1, y_2) \rightarrow (Q_1Q_2x, P_1^{-1}y_1, P_2^{-1}y_2)$, (3.11) turns out to be

$$\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{K(x, y_1, y_2)^\lambda} \right\|_{L_x^q} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}, \tag{3.12}$$

where

$$K(x, y_1, y_2) = |x_1 - y_{11}| + |x_2 - y_{12}| + |y_{13}| + |x_2 - y_{21}| + |x_3 - y_{22}| + |y_{23}|,$$

$x = (x_1, x_2, x_3)$, $y_i = (y_{i1}, y_{i2}, y_{i3})$, $x_1, y_{11} \in \mathbb{R}^{m-r_2}$, $x_2, y_{12}, y_{21} \in \mathbb{R}^{r_1+r_2-m}$, $x_3, y_{22} \in \mathbb{R}^{m-r_1}$, $y_{13} \in \mathbb{R}^{n_1-r_1}$ and $y_{23} \in \mathbb{R}^{n_2-r_2}$.

Since $r_1, r_2 < m$, similar arguments as that in Subsection 3.2 show that $q \geq p_i, i = 1, 2$. Hence $q \geq \max\{p_1, p_2\}$.

We prove the rest in two cases.

(i) $\min\{p_1, p_2\} = 1$.

We show that $q > p_2$ when $p_1 = 1, r_2 = n_2$ and $r_1 + r_2 = m$. In this case, $K(x, y_1, y_2) = |x_1 - y_{11}| + |y_{13}| + |x_3 - y_2|$ and (3.12) becomes

$$\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|x_1 - y_{11}| + |y_{13}| + |x_3 - y_2|)^\lambda} \right\|_{L_x^q} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}.$$

Setting $f_1 = (1/\delta^{n_1})\chi_{\{|y_1| \leq \delta\}}$ and letting $\delta \rightarrow 0$, we get

$$\left\| \int_{\mathbb{R}^{n_2}} \frac{f_2(y_2)dy_2}{(|x_1| + |x_3 - y_2|)^{n_2/p_2' + m/q}} \right\|_{L_x^q} \lesssim \|f_2\|_{L^{p_2}},$$

which is equivalent to

$$\left\| \int_{\mathbb{R}^m} \frac{h(x_1, x_3) dx_1 dx_3}{(|x_1| + |x_3 - y_2|)^{n_2/p'_2 + m/q}} \right\|_{L^{p'_2}_{y_2}} \lesssim \|h\|_{L^{q'}}.$$

By Theorem 2.1, we have $p'_2 > q'$. Hence $q > p_2$.

Similarly we can prove that $q > p_1$ when $p_2 = 1, r_1 = n_1$ and $r_1 + r_2 = m$.

(ii) $1 < p_1, p_2 < \infty$.

First we show that $q > p_1$ when $p_1 = p_2, r_1 + r_2 = m$ and $n_i = r_i$ for some i .

Assume on the contrary that $q = p_1 = p_2$. Since $\lambda < n_1 + n_2$, we have $(n_1, n_2) \neq (r_1, r_2)$. Hence $n_1 > r_1$ or $n_2 > r_2$. Without loss of generality, we assume that $n_1 = r_1$ and $n_2 > r_2$. In this case, $\lambda = r_1 + r_2 + (n_2 - r_2)/p'_2$. We see from (3.12) that

$$\left\| \int_{\mathbb{R}^{r_1+m}} \frac{h(x) f_1(y_1) dx dy_1}{(|x_1 - y_1| + |x_3 - y_{22}| + |y_{23}|)^\lambda} \right\|_{L^{p'_2}_{y_2}} \lesssim \|f_1\|_{L^{p_1}} \|h\|_{L^{q'}}. \tag{3.13}$$

Set $h = \chi_{\{|x| \leq 1\}}$ and $f_1 = \chi_{\{|y_1| \leq 1\}}$. For $\delta > 0$ small enough and $|y_2| \leq \delta$,

$$\begin{aligned} & \int_{\mathbb{R}^{r_1+m}} \frac{h(x) f_1(y_1) dx dy_1}{(|x_1 - y_1| + |x_3 - y_{22}| + |y_{23}|)^\lambda} \\ & \geq \int_{\substack{|y_1| \leq \delta \\ |x_1 - y_1| \leq |y_{23}| \\ |x_3 - y_{22}| \leq |y_{23}|}} \frac{h(x) f_1(y_1) dx dy_1}{(|x_1 - y_1| + |x_3 - y_{22}| + |y_{23}|)^\lambda} \\ & \gtrsim \frac{\delta^{r_1}}{|y_{23}|^{(n_2 - r_2)/p'_2}}, \end{aligned}$$

which contradicts (3.13).

Next we show that $q > p_1$ when $p_1 = p_2 > 2, m = r_1 + r_2, n_1 > r_1$ and $n_2 > r_2$.

Again, assume that $q = p_1 = p_2 > 2$. We have

$$\left\| \int_{\mathbb{R}^{r_1+m}} \frac{h(x) f_1(y_1) dx dy_1}{(|x_1 - y_{11}| + |y_{13}| + |x_3 - y_{22}| + |y_{23}|)^\lambda} \right\|_{L^{p'_2}_{y_2}} \lesssim \|f_1\|_{L^{p_1}} \|h\|_{L^{q'}}.$$

Set

$$h(x) = \chi_{\{|x| \leq 1\}}(x) \quad \text{and} \quad f_1(y_1) = \frac{\chi_{\{|y_1| \leq 1/2\}}(y_1)}{|y_{13}|^{(n_1 - r_1)/p_1} (\log 1/|y_{13}|)^{(1+\varepsilon)/p_1}}.$$

For $\delta > 0$ small enough and $|y_2| \leq \delta$,

$$\begin{aligned} & \int_{\mathbb{R}^{n_1+m}} \frac{h(x)f_1(y_1)dxdy_1}{(|x_1 - y_{11}| + |y_{13}| + |x_3 - y_{22}| + |y_{23}|)^\lambda} \\ & \geq \int_{\substack{|y_{11}| \leq \delta \\ |x_1 - y_{11}| \leq |y_{23}| \\ |y_{23}|^2 \leq |y_{13}| \leq |y_{23}| \\ |x_3 - y_{22}| \leq |y_{23}|}} \frac{h(x)f_1(y_1)dxdy_1}{(|x_1 - y_{11}| + |y_{13}| + |x_3 - y_{22}| + |y_{23}|)^\lambda} \\ & \quad \times \frac{1}{|y_{13}|^{(n_1-r_1)/p_1}(\log 1/|y_{13}|)^{(1+\varepsilon)/p_1}} \\ & \gtrsim \frac{\delta^{r_1}}{|y_{23}|^{(n_2-r_2)/p_2'}(\log 1/|y_{23}|)^{(1+\varepsilon)/p_1}}. \end{aligned}$$

Since $p_1 = p_2 > 2$, then there is some $\varepsilon > 0$ such that $p_2'(1 + \varepsilon)/p_1 < 1$. Hence

$$\left\| \int_{\mathbb{R}^{n_1+m}} \frac{h(x)f_1(y_1)dxdy_1}{(|x_1 - y_{11}| + |y_{13}| + |x_3 - y_{22}| + |y_{23}|)^\lambda} \right\|_{L^{p_2'}} = \infty.$$

This completes the proof of the necessity.

4 Proof of Theorem 1.1: the sufficiency

In this section, we give the proof of the sufficiency part in Theorem 1.1.

First, we consider the case $q = \infty$. In this case, $1 < p_1, p_2 < \infty$ and $1/p_1 + 1/p_2 \geq 1$. Note that

$$\begin{aligned} I_{\lambda,D}(f_1, f_2)(x) &= \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|D_1x - y_1| + |D_2x - y_2|)^\lambda} \\ &= \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1 + D_1x)f_2(y_2 + D_2x)dy_1dy_2}{(|y_1| + |y_2|)^\lambda}. \end{aligned}$$

By Lemma 2.1, we have

$$|I_{\lambda,D}(f_1, f_2)| \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}, \quad \forall x \in \mathbb{R}^m.$$

Hence $I_{\lambda,D}$ is bounded from $L^{p_1} \times L^{p_2}$ to L^q . For the case $q < \infty$, we split the proof in several subsections.

4.1 The case $r_1 = r_2 = m$

As in Subsection 3.1, we only need to prove that for $f_1 \in L^{p_1}$ and $f_2 \in L^{p_2}$,

$$\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_{11}, y_{12})f_2(y_{21}, y_{22})dy_1dy_2}{(|x - y_{11}| + |x - y_{21}| + |y_{12}| + |y_{22}|)^\lambda} \right\|_{L^q} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}, \quad (4.1)$$

where $y_i = (y_{i1}, y_{i2})$, $x, y_{i1} \in \mathbb{R}^m$, $y_{i2} \in \mathbb{R}^{n_i-m}$, $i = 1, 2$. There are two subcases.

(A1) $1 < p_1, p_2 < \infty$. We prove the conclusion in two subcases.

(A1a) $0 < 1/q < 1/p_1 + 1/p_2$. Choose q_1, q_2 such that

$$\frac{1}{q_1} = \frac{1}{q} \cdot \frac{1/p_1}{1/p_1 + 1/p_2} \quad \text{and} \quad \frac{1}{q_2} = \frac{1}{q} \cdot \frac{1/p_2}{1/p_1 + 1/p_2}.$$

Then $1/q = 1/q_1 + 1/q_2$ and $1 < p_i < q_i < \infty$. Let

$$\lambda_i := \frac{n_i}{p'_i} + \frac{m}{q_i}, \quad i = 1, 2.$$

We have $\lambda = \lambda_1 + \lambda_2$ and $0 < \lambda_i < n_i$. By Theorem 2.1,

$$\left\| \int_{\mathbb{R}^{n_i}} \frac{f_i(y_i) dy_i}{(|x - y_{i1}| + |y_{i2}|)^{\lambda_i}} \right\|_{L^{q_i}} \lesssim \|f_i\|_{L^{p_i}}.$$

Hence

$$\begin{aligned} & \left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1) f_2(y_2) dy_1 dy_2}{(|x - y_{11}| + |y_{12}| + |x - y_{21}| + |y_{22}|)^\lambda} \right\|_{L^q_x} \\ & \leq \left\| \int_{\mathbb{R}^{n_1}} \frac{f_1(y_1) dy_1}{(|x - y_{11}| + |y_{12}|)^{\lambda_1}} \int_{\mathbb{R}^{n_2}} \frac{f_2(y_2) dy_2}{(|x - y_{21}| + |y_{22}|)^{\lambda_2}} \right\|_{L^q_x} \\ & \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}. \end{aligned}$$

(A1b) $1/q = 1/p_1 + 1/p_2$. In this case, $n_1, n_2 > m$ and $1/p_1 + 1/p_2 \geq 1$. Let us prove (4.1).

For any $f_1 \in L^{p_1}$ and $f_2 \in L^{p_2}$, we have

$$\begin{aligned} & \int_{\mathbb{R}^{n_1+n_2}} \frac{|f_1(y_{11}, y_{12}) f_2(y_{21}, y_{22})| dy_1 dy_2}{(|x - y_{11}| + |x - y_{21}| + |y_{12}| + |y_{22}|)^\lambda} \\ & \lesssim \int_{\mathbb{R}^{n_1+n_2-2m}} \frac{M_x f_1(x, y_{12}) M_x f_2(x, y_{22}) dy_{12} dy_{22}}{(|y_{12}| + |y_{22}|)^{(n_1-m)/p'_1 + (n_2-m)/p'_2}}, \end{aligned}$$

where

$$M_x f_1(x, y_{12}) = \sup_{r>0} \frac{1}{r^m} \int_{|y_{11}-x|\leq r} |f_1(y_{11}, y_{12})| dy_{11}$$

is the partially maximal function of f_1 , and $M_x f_2(x, y_{22})$ is defined similarly. It follows from Lemma 2.1 that

$$\int_{\mathbb{R}^{n_1+n_2}} \frac{|f_1(y_{11}, y_{12}) f_2(y_{21}, y_{22})| dy_1 dy_2}{(|x - y_{11}| + |x - y_{21}| + |y_{12}| + |y_{22}|)^\lambda} \lesssim \|M_x f_1(x, \cdot)\|_{L^{p_1}} \|M_x f_2(x, \cdot)\|_{L^{p_2}}.$$

Since $1/q = 1/p_1 + 1/p_2$, by Hölder's inequality, we get

$$\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_{11}, y_{12})f_2(y_{21}, y_{22})dy_1dy_2}{(|x - y_{11}| + |x - y_{21}| + |y_{12}| + |y_{22}|)^\lambda} \right\|_{L_x^q} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}.$$

(A2) $\#\{i : 1 < p_i < \infty\} = 1$. Without loss of generality, assume that $1 < p_2 < \infty$. There are two subcases.

(A2a) $p_1 = 1$. In this case, $1 < p_2 \leq q < \infty$. (4.1) is equivalent to

$$\left\| \int_{\mathbb{R}^{n_2+m}} \frac{f_2(y_{21}, y_{22})h(x)dx dy_2}{(|x - y_{11}| + |x - y_{21}| + |y_{12}| + |y_{22}|)^{n_2/p_2'+m/q}} \right\|_{L_{y_1}^\infty} \lesssim \|f_2\|_{L^{p_2}} \|h\|_{L^{q'}}. \quad (4.2)$$

Note that for any $y_1 \in \mathbb{R}^{n_1}$,

$$\begin{aligned} & \int_{\mathbb{R}^{n_2+m}} \frac{|f_2(y_{21}, y_{22})h(x)|dx dy_2}{(|x - y_{11}| + |x - y_{21}| + |y_{12}| + |y_{22}|)^{n_2/p_2'+m/q}} \\ & \lesssim \int_{\mathbb{R}^{2m}} \frac{\|f_2(y_{21}, \cdot)\|_{L^{p_2}} |h(x)|dx dy_{21}}{(|x - y_{11}| + |x - y_{21}|)^{m/p_2'+m/q}} \\ & \approx \int_{\mathbb{R}^{2m}} \frac{\|f_2(y_{21}, \cdot)\|_{L^{p_2}} |h(x)|dx dy_{21}}{(|x - y_{11}| + |y_{11} - y_{21}|)^{m/p_2'+m/q}} \\ & = \int_{\mathbb{R}^{2m}} \frac{\|f_2(y_{21} + y_{11}, \cdot)\|_{L^{p_2}} |h(x + y_{11})|dx dy_{21}}{(|x| + |y_{21}|)^{m/p_2'+m/q}} \\ & \lesssim \|f_2(\cdot + y_{11}, \cdot)\|_{L^{p_2}} \|h(\cdot + y_{11})\|_{L^{q'}}, \end{aligned}$$

where we use Lemma 2.1 in the last step. Hence (4.2) is true.

(A2b) $p_1 = \infty$. In this case, $p_2 < q < \infty$ and $\lambda = n_1 + n_2/p' + m/q$. We have

$$\begin{aligned} & \int_{\mathbb{R}^{n_1+n_2}} \frac{|f_1(y_{11}, y_{12})f_2(y_{21}, y_{22})|dy_1dy_2}{(|x - y_{11}| + |x - y_{21}| + |y_{12}| + |y_{22}|)^\lambda} \\ & \lesssim \|f_1\|_{L^\infty} \int_{\mathbb{R}^{n_2}} \frac{|f_2(y_{21}, y_{22})|dy_2}{(|x - y_{21}| + |y_{22}|)^{n_2/p'+m/q}}. \end{aligned}$$

Now the conclusion follows from Theorem 2.1.

4.2 The case $r_1 = 0$ and $r_2 = m$

In this case, we need to show that

$$\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|y_1| + |x - y_{21}| + |y_{22}|)^\lambda} \right\|_{L_x^q} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}, \quad (4.3)$$

where $y_2 = (y_{21}, y_{22})$, $y_{21} \in \mathbb{R}^m$ and $y_{22} \in \mathbb{R}^{n_2-m}$. We prove the conclusion in three subcases.

(B1) $p_2 = 1$. In this case, $1 < p_1 \leq q < \infty$. It suffices to show that

$$\left\| \int_{\mathbb{R}^{n_1+m}} \frac{f_1(y_1)h(x)dxdy_1}{(|y_1| + |x - y_{21}| + |y_{22}|)^\lambda} \right\|_{L_{y_2}^\infty} \lesssim \|f_1\|_{L^{p_1}} \|h\|_{L^{q'}}. \tag{4.4}$$

Observe that

$$\begin{aligned} \int_{\mathbb{R}^{n_1+m}} \frac{|f_1(y_1)h(x)|dxdy_1}{(|y_1| + |x - y_{21}| + |y_{22}|)^\lambda} &= \int_{\mathbb{R}^{n_1+m}} \frac{|f_1(y_1)h(x + y_{21})|dxdy_1}{(|y_1| + |x| + |y_{22}|)^{n_1'/p_1+m/q}} \\ &\leq \int_{\mathbb{R}^{n_1+m}} \frac{|f_1(y_1)h(x + y_{21})|dxdy_1}{(|y_1| + |x|)^{n_1'/p_1+m/q}}. \end{aligned}$$

By Lemma 2.1, we have

$$\int_{\mathbb{R}^{n_1+m}} \frac{|f_1(y_1)h(x)|dxdy_1}{(|y_1| + |x - y_{21}| + |y_{22}|)^\lambda} \lesssim \|f_1\|_{L^{p_1}} \|h\|_{L^{q'}}, \quad \forall y_2 \in \mathbb{R}^{n_2}.$$

Hence (4.4) is true.

(B2) $p_2 > 1$ and $p_2 < q < \infty$. By Hölder's inequality, we have

$$\int_{\mathbb{R}^{n_1+n_2}} \frac{|f_1(y_1)f_2(y_2)|dy_1dy_2}{(|y_1| + |x - y_{21}| + |y_{22}|)^\lambda} \lesssim \int_{\mathbb{R}^{n_2}} \frac{\|f_1\|_{L^{p_1}}|f_2(y_2)|dy_2}{(|x - y_{21}| + |y_{22}|)^{n_2/p_2'+m/q}}.$$

Now we see from Theorem 2.1 that (4.3) is true.

(B3) $p_2 > 1$ and $q = p_2$. In this case, $n_2 > m$, $1 < p_1, p_2 < \infty$ and $1/p_1 + 1/p_2 \geq 1$. By Minkowski's and Young's inequalities, we have

$$\begin{aligned} &\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|y_1| + |x - y_{21}| + |y_{22}|)^\lambda} \right\|_{L_x^q} \\ &\lesssim \int_{\mathbb{R}^{n_1+n_2-m}} \frac{f_1(y_1)\|f_2(\cdot, y_{22})\|_{L^{p_2}}dy_1dy_{22}}{(|y_1| + |y_{22}|)^{n_1/p_1+(n_2-m)/p_2'}}. \end{aligned}$$

Now it follows from Lemma 2.1 that

$$\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|y_1| + |x - y_{21}| + |y_{22}|)^\lambda} \right\|_{L_x^q} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}.$$

4.3 The case $0 < r_1 < r_2 = m$ or $0 < r_2 < r_1 = m$

We consider only the case $0 < r_1 < r_2 = m$. As in the necessity part, it suffices to show that

$$\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|x_1 - y_{11}| + |y_{12}| + |x - y_{21}| + |y_{22}|)^\lambda} \right\|_{L_x^q} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}, \tag{4.5}$$

where $x = (x_1, x_2)$, $x_1 \in \mathbb{R}^{r_1}$, $x_2 \in \mathbb{R}^{m-r_1}$, $y_{i1} \in \mathbb{R}^{r_i}$ and $y_{i2} \in \mathbb{R}^{n_i-r_i}$, $i = 1, 2$.

We prove the conclusion in four subcases.

(C1) $p_2 = 1$. In this case, $1 < p_1 \leq q < \infty$. (4.5) is equivalent to

$$\left\| \int_{\mathbb{R}^{n_1+m}} \frac{f_1(y_1)h(x)dxdy_1}{(|x_1 - y_{11}| + |y_{12}| + |x - y_{21}| + |y_{22}|)^\lambda} \right\|_{L_{y_2}^\infty} \lesssim \|f_1\|_{L^{p_1}} \|h\|_{L^{q'}}. \tag{4.6}$$

Observe that

$$\begin{aligned} & \int_{\mathbb{R}^{n_1+m}} \frac{|f_1(y_1)h(x)|dxdy_1}{(|x_1 - y_{11}| + |y_{12}| + |x - y_{21}| + |y_{22}|)^\lambda} \\ &= \int_{\mathbb{R}^{n_1+m}} \frac{|f_1(y_1 + (y_{21}^{(1)}, \dots, y_{21}^{(r_1)}, 0, \dots, 0))h(x + y_{21})|dxdy_1}{(|x_1 - y_{11}| + |y_{12}| + |x| + |y_{22}|)^{n_1'/p_1+m/q}} \\ &\lesssim \int_{\mathbb{R}^{n_1+m}} \frac{|f_1(y_1 + (y_{21}^{(1)}, \dots, y_{21}^{(r_1)}, 0, \dots, 0))h(x + y_{21})|dxdy_1}{(|y_1| + |x|)^{n_1'/p_1+m/q}}. \end{aligned}$$

By Lemma 2.1, we have

$$\int_{\mathbb{R}^{n_1+m}} \frac{|f_1(y_1)h(x)|dxdy_1}{(|x_1 - y_{11}| + |y_{12}| + |x - y_{21}| + |y_{22}|)^\lambda} \lesssim \|f_1\|_{L^{p_1}} \|h\|_{L^{q'}}, \forall y_2 \in \mathbb{R}^{n_2}.$$

Hence (4.6) is true.

(C2) $p_1 = 1$. In this case, $1 < p_2 \leq q < \infty$. It suffices to show that for any $f_2 \in L^2$ and $h \in L^{q'}$,

$$\left| \int_{\mathbb{R}^{n_2+m}} \frac{f_2(y_2)h(x)dxdy_2}{(|x_1 - y_{11}| + |y_{12}| + |x - y_{21}| + |y_{22}|)^\lambda} \right| \lesssim \|f_2\|_{L^{p_2}} \|h\|_{L^{q'}}, \quad \forall y_1 \in \mathbb{R}^{n_1}. \tag{4.7}$$

Denote $y_{21} = (y_{211}, y_{212})$, where $y_{211} \in \mathbb{R}^{r_1}$, $y_{212} \in \mathbb{R}^{m-r_1}$. We rewrite the above inequality as

$$\left| \int_{\mathbb{R}^{n_2+m}} \frac{f_2(y_2)h(x)dxdy_2}{(|x_1 - y_{11}| + |y_{12}| + |x_1 - y_{211}| + |x_2 - y_{212}| + |y_{22}|)^\lambda} \right| \lesssim \|f_2\|_{L^{p_2}} \|h\|_{L^{q'}}.$$

Using Young's inequality when computing the integration with respect to $dx_2 dy_{212}$, we get

$$\begin{aligned} & \left| \int_{\mathbb{R}^{n_2+m}} \frac{f_2(y_2)h(x)dxdy_2}{(|x_1 - y_{11}| + |y_{12}| + |x_1 - y_{211}| + |x_2 - y_{212}| + |y_{22}|)^\lambda} \right| \\ &\lesssim \int_{\mathbb{R}^{2r_1+n_2-m}} \frac{\|f_2(y_{211}, \cdot, y_{22})\|_{L_{y_{212}}^{p_2}} \|h(x_1, \cdot)\|_{L_{x_2}^{q'}} dx_1 dy_{211} dy_{22}}{(|x_1 - y_{11}| + |y_{12}| + |x_1 - y_{211}| + |y_{22}|)^{(n_2+r_1-m)/p_2+r_1/q}} \\ &= \int_{\mathbb{R}^{2r_1+n_2-m}} \frac{\|f_2(y_{211} + y_{11}, \cdot, y_{22})\|_{L_{y_{212}}^{p_2}} \|h(x_1 + y_{11}, \cdot)\|_{L_{x_2}^{q'}} dx_1 dy_{211} dy_{22}}{(|x_1| + |y_{12}| + |x_1 - y_{211}| + |y_{22}|)^{(n_2+r_1-m)/p_2+r_1/q}} \\ &\lesssim \int_{\mathbb{R}^{2r_1+n_2-m}} \frac{\|f_2(y_{211} + y_{11}, \cdot, y_{22})\|_{L_{y_{212}}^{p_2}} \|h(x_1 + y_{11}, \cdot)\|_{L_{x_2}^{q'}} dx_1 dy_{211} dy_{22}}{(|x_1| + |y_{211}| + |y_{22}|)^{(n_2+r_1-m)/p_2+r_1/q}} \\ &\lesssim \|f_2\|_{L^{p_2}} \|h\|_{L^{q'}}, \end{aligned}$$

where we use Lemma 2.1 in the last step. Hence (4.7) is true.

(C3) $p_1 = \infty$. In this case, $1 < p_2 < q < \infty$. We have

$$\begin{aligned} & \left| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|x_1 - y_{11}| + |y_{12}| + |x - y_{21}| + |y_{22}|)^\lambda} \right| \\ & \lesssim \int_{\mathbb{R}^{n_1}} \frac{\|f_1\|_{L^\infty} f_2(y_2)dy_2}{(|x - y_{21}| + |y_{22}|)^{\lambda-n_1}}. \end{aligned}$$

Now the conclusion follows from Theorem 2.1.

(C4) $1 < p_1, p_2 < \infty$ and $p_2 \leq q < \infty$. Using Hölder’s inequality when computing the integration with respect to y_{12} and y_{22} , we get

$$\begin{aligned} & \left| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|x_1 - y_{11}| + |y_{12}| + |x - y_{21}| + |y_{22}|)^\lambda} \right| \\ & \lesssim \int_{\mathbb{R}^{r_1+m}} \frac{\|f_1(y_{11}, \cdot)\|_{L^{p_1}} \|f_2(y_{21}, \cdot)\|_{L^{p_2}} dy_{11} dy_{21}}{(|x_1 - y_{11}| + |x - y_{21}|)^{r_1/p_1+m/p_2+m/q}}. \end{aligned}$$

Denote $y_{21} = (y_{211}, y_{212})$, where $y_{211} \in \mathbb{R}^{r_1}$ and $y_{212} \in \mathbb{R}^{m-r_1}$. By Young’s inequality, we get

$$\begin{aligned} & \left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{(|x_1 - y_{11}| + |y_{12}| + |x - y_{21}| + |y_{22}|)^\lambda} \right\|_{L^q_x} \\ & \lesssim \int_{\mathbb{R}^{2r_1}} \frac{\|f_1(y_{11}, \cdot)\|_{L^{p_1}} \|f_2(y_{211}, \cdot)\|_{L^{p_2}} dy_{11} dy_{211}}{(|x_1 - y_{11}| + |x_1 - y_{211}|)^{r_1/p_1+r_1/p_2+r_1/q}}. \end{aligned}$$

Now the conclusion follows from Proposition 1.1.

4.4 The case $0 < r_1, r_2 < m$

As in the necessity part, we only need to show that

$$\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{K(x, y_1, y_2)^\lambda} \right\|_{L^q_x} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}},$$

where

$$K(x, y_1, y_2) = |x_1 - y_{11}| + |x_2 - y_{12}| + |y_{13}| + |x_2 - y_{21}| + |x_3 - y_{22}| + |y_{23}|,$$

$x = (x_1, x_2, x_3)$, $y_i = (y_{i1}, y_{i2}, y_{i3})$, $x_1, y_{11} \in \mathbb{R}^{m-r_2}$, $x_2, y_{12}, y_{21} \in \mathbb{R}^{r_1+r_2-m}$, $x_3, y_{22} \in \mathbb{R}^{m-r_1}$, $y_{13} \in \mathbb{R}^{n_1-r_1}$ and $y_{23} \in \mathbb{R}^{n_2-r_2}$.

There are five subcases.

(D1) $p_1 = 1$ and $r_2 < n_2$. In this case, $p_2 \leq q < \infty$. Observe that

$$K(x, y_1, y_2) \geq |x_1 - y_{11}| + |x_2 - y_{21}| + |x_3 - y_{22}| + |y_{23}|.$$

We have

$$\begin{aligned} & \left\| \int_{\mathbb{R}^{n_2}} \frac{f_2(y_2)dy_2}{K(x, y_1, y_2)^\lambda} \right\|_{L^q_{(x_2, x_3)}} \\ & \leq \left\| \int_{\mathbb{R}^{n_2}} \frac{f_2(y_2)dy_2}{(|x_1 - y_{11}| + |x_2 - y_{21}| + |x_3 - y_{22}| + |y_{23}|)^\lambda} \right\|_{L^q_{(x_2, x_3)}} \\ & \lesssim \int_{\mathbb{R}^{n_2-r_2}} \frac{\|f_2(\cdot, y_{23})\|_{L^{p_2}} dy_{23}}{(|x_1 - y_{11}| + |y_{23}|)^{(n_2-r_2)/p_2' + (m-r_2)/q}}, \end{aligned}$$

where we use Young’s inequality in the last step. Hence

$$\left\| \int_{\mathbb{R}^{n_2}} \frac{f_2(y_2)dy_2}{K(x, y_1, y_2)^\lambda} \right\|_{L^q_x} \lesssim \left\| \int_{\mathbb{R}^{n_2-r_2}} \frac{\|f_2(\cdot, y_{23})\|_{L^{p_2}} dy_{23}}{(|x_1 - y_{11}| + |y_{23}|)^{(n_2-r_2)/p_2' + (m-r_2)/q}} \right\|_{L^q_{x_1}}.$$

By Lemma 2.1,

$$\left\| \int_{\mathbb{R}^{n_2}} \frac{f_2(y_2)dy_2}{K(x, y_1, y_2)^\lambda} \right\|_{L^q_x} \lesssim \|f_2\|_{L^{p_2}}, \quad \forall y_1 \in \mathbb{R}^{n_1}.$$

It follows from Minkowski’s inequality that

$$\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{K(x, y_1, y_2)^\lambda} \right\|_{L^q_x} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}. \tag{4.8}$$

(D2) $p_1 = 1, r_2 = n_2$ and $r_1 + r_2 > m$. In this case, $p_2 \leq q < \infty$ and

$$\begin{aligned} K(x, y_1, y_2) &= |x_1 - y_{11}| + |x_2 - y_{12}| + |y_{13}| + |x_2 - y_{21}| + |x_3 - y_{22}| \\ &\geq |x_1 - y_{11}| + |x_2 - y_{12}| + |x_2 - y_{21}| + |x_3 - y_{22}| \\ &\approx |x_1 - y_{11}| + |x_2 - y_{12}| + |y_{12} - y_{21}| + |x_3 - y_{22}|. \end{aligned}$$

Hence

$$\begin{aligned} & \left\| \int_{\mathbb{R}^{n_2}} \frac{f_2(y_2)dy_2}{K(x, y_1, y_2)^\lambda} \right\|_{L^q_{x_3}} \\ & \leq \left\| \int_{\mathbb{R}^{n_2}} \frac{f_2(y_2)dy_2}{(|x_1 - y_{11}| + |x_2 - y_{12}| + |x_3 - y_{22}| + |y_{12} - y_{21}|)^\lambda} \right\|_{L^q_{x_3}} \\ & \lesssim \int_{\mathbb{R}^{n_2+r_1-m}} \frac{\|f_2(y_{21}, \cdot)\|_{L^{p_2}} dy_{21}}{(|x_1 - y_{11}| + |x_2 - y_{12}| + |y_{12} - y_{21}|)^{(n_2+r_1-m)/p_2' + r_1/q}} \\ & = \int_{\mathbb{R}^{n_2+r_1-m}} \frac{\|f_2(y_{21} + y_{12}, \cdot)\|_{L^{p_2}} dy_{21}}{(|x_1 - y_{11}| + |x_2 - y_{12}| + |y_{21}|)^{(n_2+r_1-m)/p_2' + r_1/q}}. \end{aligned}$$

It follows from Lemma 2.1 that

$$\left\| \int_{\mathbb{R}^{n_2}} \frac{f_2(y_2)dy_2}{K(x, y_1, y_2)^\lambda} \right\|_{L_x^q} \lesssim \|f_2\|_{L^{p_2}}, \quad \forall y_1 \in \mathbb{R}^{n_1}.$$

Thus (4.8) is true.

(D3) $p_1 = 1, r_2 = n_2$ and $r_1 + r_2 = m$. In this case, $p_2 < q < \infty$ and

$$K(x, y_1, y_2) = |x_1 - y_{11}| + |y_{13}| + |x_3 - y_2| \geq |x_1 - y_{11}| + |x_3 - y_2|.$$

Hence

$$\int_{\mathbb{R}^{n_2}} \frac{|f_2(y_2)|dy_2}{K(x, y_1, y_2)^\lambda} \leq \int_{\mathbb{R}^{n_2}} \frac{|f_2(y_2)|dy_2}{(|x_1 - y_{11}| + |x_3 - y_2|)^\lambda}.$$

It follows from Minkowski's inequality that

$$\left\| \int_{\mathbb{R}^{n_2}} \frac{f_2(y_2)dy_2}{K(x, y_1, y_2)^\lambda} \right\|_{L_{x_1}^q} \leq \int_{\mathbb{R}^{n_2}} \frac{|f_2(y_2)|dy_2}{|x_3 - y_2|^{r_2/p_2' + r_2/q}}.$$

By Theorem 2.1, we get

$$\left\| \int_{\mathbb{R}^{n_2}} \frac{|f_2(y_2)|dy_2}{|x_3 - y_2|^{r_2/p_2' + r_2/q}} \right\|_{L_{x_3}^q} \lesssim \|f_2\|_{L^{p_2}}.$$

Hence

$$\left\| \int_{\mathbb{R}^{n_2}} \frac{f_2(y_2)dy_2}{K(x, y_1, y_2)^\lambda} \right\|_{L_x^q} = \left\| \left\| \int_{\mathbb{R}^{n_2}} \frac{f_2(y_2)dy_2}{K(x, y_1, y_2)^\lambda} \right\|_{L_{x_1}^q} \right\|_{L_{x_3}^q} \lesssim \|f_2\|_{L^{p_2}}, \quad \forall y_1 \in \mathbb{R}^{n_1}.$$

Therefore, (4.8) is true.

(D4) $p_2 = 1$. Similarly to Cases (D1)–(D3) we get the conclusion.

(D5) $1 < p_1, p_2 < \infty$ and $\max\{p_1, p_2\} \leq q < \infty$. There are two subcases.

(D5a) $r_1 + r_2 > m$. Recall that

$$K(x, y_1, y_2) = |x_1 - y_{11}| + |x_2 - y_{12}| + |y_{13}| + |x_2 - y_{21}| + |x_3 - y_{22}| + |y_{23}|.$$

Using Hölder's inequality when computing the integrations dy_{13} and dy_{23} , we get

$$\begin{aligned} & \left| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{K(x, y_1, y_2)^\lambda} \right| \\ & \lesssim \int_{\mathbb{R}^{n_1+n_2}} \frac{\|f_1(y_{11}, y_{12}, \cdot)\|_{L^{p_1}} \|f_2(y_{21}, y_{22}, \cdot)\|_{L^{p_2}} dy_{11}dy_{12}dy_{21}dy_{22}}{(|x_1 - y_{11}| + |x_2 - y_{12}| + |x_2 - y_{21}| + |x_3 - y_{22}|)^{r_1/p_1' + r_2/p_2' + m/q}}. \end{aligned} \tag{4.9}$$

We see from Young's inequality that

$$\begin{aligned} & \left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{K(x, y_1, y_2)^\lambda} \right\|_{L^q_{(x_1, x_3)}} \\ & \lesssim \int_{\mathbb{R}^{2(r_1+r_2-m)}} \frac{\|f_1(\cdot, y_{12}, \cdot)\|_{L^{p_1}} \|f_2(y_{21}, \cdot)\|_{L^{p_2}} dy_{12}dy_{21}}{(|x_2 - y_{12}| + |x_2 - y_{21}|)^{(r_1+r_2-m)(1/p'_1+1/p'_2+1/q)}}. \end{aligned}$$

By Proposition 1.1, we get the conclusion as desired.

(D5b) $r_1 + r_2 = m$. In this case, the variables x_2, y_{12}, y_{21} do not exist. If $p_1 < p_2$, we see from Hölder's and Young's inequalities that

$$\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{K(x, y_1, y_2)^\lambda} \right\|_{L^q_{x_3}} \lesssim \int_{\mathbb{R}^{n_1}} \frac{|f_1(y_1)| \cdot \|f_2\|_{L^{p_2}} dy_1}{(|x_1 - y_{11}| + |y_{13}|)^{n_1/p'_1+r_1/q}}.$$

By Theorem 2.1, we get

$$\left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{K(x, y_1, y_2)^\lambda} \right\|_{L^q_x} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}.$$

If $p_2 < p_1$ or $p_1 = p_2 < q$, with similar arguments we get the conclusion.

If $q = p_1 = p_2$, then we have $q \leq 2$, $n_1 > r_1$ and $n_2 > r_2$. By Young's inequality, we get

$$\begin{aligned} & \left\| \int_{\mathbb{R}^{n_1+n_2}} \frac{f_1(y_1)f_2(y_2)dy_1dy_2}{K(x, y_1, y_2)^\lambda} \right\|_{L^q_x} \\ & \lesssim \int_{\mathbb{R}^{n_1+n_2-r_1-r_2}} \frac{\|f_1(\cdot, y_{13})\|_{L^{p_1}} \|f_2(\cdot, y_{23})\|_{L^{p_2}} dy_{13}dy_{23}}{(|y_{13}| + |y_{23}|)^{(n_1-r_1)/p'_1+(n_2-r_2)/p'_2}}. \end{aligned}$$

Now the conclusion follows from Lemma 2.1. This completes the proof. \square

Acknowledgements

This work was partially supported by the National Natural Science Foundation of China (Nos. 12271267, 12171250 and U21A20426) and the Fundamental Research Funds for the Central Universities. The authors thank the referees very much for valuable suggestions.

References

- [1] A. Bényi, W. Damián, K. Moen, and R. H. Torres, Compactness properties of commutators of bilinear fractional integrals, *Math. Z.*, 280(1-2) (2015), 569–582.
- [2] M. Cao and Q. Xue, A revisit on commutators of linear and bilinear fractional integral operator, *Tohoku Math. J. (2)*, 71(2) (2019), 303–318.

- [3] L. Chaffee and R. H. Torres, Characterization of compactness of the commutators of bilinear fractional integral operators, *Potential Anal.*, 43(3) (2015), 481–494.
- [4] J. Chen and D. Fan, A bilinear fractional integral on compact Lie groups, *Canad. Math. Bull.*, 54(2) (2011), 207–216.
- [5] T. Chen and W. Sun, Extension of multilinear fractional integral operators to linear operators on mixed-norm Lebesgue spaces, *Math. Ann.*, 379(3-4) (2021), 1089–1172.
- [6] T. Chen and W. Sun, Hardy–Littlewood–Sobolev inequality on mixed-norm Lebesgue spaces, *J. Geom. Anal.*, 32 (2022), 101.
- [7] M. Christ and J.-L. Journé, Polynomial growth estimates for multilinear singular integral operators, *Acta Math.*, 159(1-2) (1987), 51–80.
- [8] R. R. Coifman and Y. Meyer, On commutators of singular integrals and bilinear singular integrals, *Trans. Amer. Math. Soc.*, 212 (1975), 315–331.
- [9] P. Germain, The second iterate for the Navier-Stokes equation, *J. Funct. Anal.*, 255(9) (2008), 2248–2264.
- [10] L. Grafakos, On multilinear fractional integrals, *Studia Math.*, 102(1) (1992), 49–56.
- [11] L. Grafakos, *Classical Fourier Analysis*, volume 249 of Graduate Texts in Mathematics, Springer, New York, third edition, 2014.
- [12] L. Grafakos, *Modern Fourier Analysis*, volume 250 of Graduate Texts in Mathematics, Springer, New York, third edition, 2014.
- [13] L. Grafakos and N. Kalton, Some remarks on multilinear maps and interpolation, *Math. Ann.*, 319(1) (2001), 151–180.
- [14] L. Grafakos and R. G. Lynch, Off-diagonal multilinear interpolation between adjoint operators, *Comment. Math.*, 55(1) (2015), 17–22.
- [15] L. Grafakos and R. H. Torres, Multilinear Calderón-Zygmund theory, *Adv. Math.*, 165(1) (2002), 124–164.
- [16] C. Hoang and K. Moen, Weighted estimates for bilinear fractional integral operators and their commutators, *Indiana Univ. Math. J.*, 67(1) (2018), 397–428.
- [17] C. E. Kenig and E. M. Stein, Multilinear estimates and fractional integration, *Math. Res. Lett.*, 6(1) (1999), 1–15.
- [18] Y. Komori-Furuya, Notes on endpoint estimates for multilinear fractional integral operators, *Proc. Amer. Math. Soc.*, 145(11) (2017), 1015–1526.
- [19] Y. Komori-Furuya, Weighted estimates for bilinear fractional integral operators: a necessary and sufficient condition for power weights, *Collect. Math.*, 71(1) (2020), 25–37.
- [20] M. Lacey and C. Thiele, L^p estimates on the bilinear Hilbert transform for $2 < p < \infty$, *Ann. of Math. (2)*, 146(3) (1997), 693–724.
- [21] M. Lacey and C. Thiele, On Calderón’s conjecture, *Ann. of Math. (2)*, 149(2) (1999), 475–496.
- [22] J. Li and P. Liu, Bilinear fractional integral along homogeneous curves, *J. Fourier Anal. Appl.*, 23(6) (2017), 1465–1479.
- [23] K. Li and W. Sun, Two weight norm inequalities for the bilinear fractional integrals, *Manuscripta Math.*, 150(1-2) (2016), 159–175.
- [24] K. Moen, New weighted estimates for bilinear fractional integral operators, *Trans. Amer. Math. Soc.*, 366(2) (2014), 627–646.
- [25] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.