Analysis in Theory and Applications
Anal. Theory Appl., Vol. **32**, No. 1 (2016), pp. 78-89

DOI: 10.4208/ata.2016.v32.n1.7

Hardy Type Estimates for Riesz Transforms Associated with Schrödinger Operators on the Heisenberg Group

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Received 24 July 2015; Accepted (in revised version) 18 December 2015

Abstract. Let $\mathbb{H}^n$ be the Heisenberg group and $Q=2n+2$ be its homogeneous dimension. In this paper, we consider the Schrödinger operator $-\Delta_{\mathbb{H}^n}+V$, where $\Delta_{\mathbb{H}^n}$ is the sub-Laplacian and V is the nonnegative potential belonging to the reverse Hölder class B_{q_1} for $q_1\geq Q/2$. We show that the operators $T_1=V(-\Delta_{\mathbb{H}^n}+V)^{-1}$ and $T_2=V^{1/2}(-\Delta_{\mathbb{H}^n}+V)^{-1/2}$ are both bounded from $H^1_L(\mathbb{H}^n)$ into $L^1(\mathbb{H}^n)$. Our results are also valid on the stratified Lie group.

Key Words: Heisenberg group, stratified Lie group, reverse Hölder class, Riesz transform, Schrödinger operator.

AMS Subject Classifications: 52B10, 65D18, 68U05, 68U07

1 Introduction

Let $L=-\Delta_{\mathbb{H}^n}+V$ be a Schrödinger operator, where $\Delta_{\mathbb{H}^n}$ is the sub-Laplacian on the Heisenberg group $\mathbb{H}^n$ and V the nonnegative potential belonging to the reverse Hölder class B_{q_1} for some $q_1\geq Q/2$ and $Q>5$. In this paper we consider the Riesz transforms associated with the Schrödinger operator L

$$T_1=V(-\Delta_{\mathbb{H}^n}+V)^{-1},\quad T_2=V^{1/2}(-\Delta_{\mathbb{H}^n}+V)^{-1/2},\quad T_3=\nabla_{\mathbb{H}^n}(-\Delta_{\mathbb{H}^n}+V)^{-1/2}.$$

We are interested in the Hardy type estimates for the Riesz transform T_i, i=1,2,3. In recent years, some problems related to Schrödinger operators and Schrödinger type operators on the Heisenberg group and other nilpotent Lie group have been investigated by a number of scholars (see [2,3,5–10,12]). Among these papers the core problem is the research of estimates for Riesz transforms associated with the Schrödinger operator L. As we know, C. C. Lin, H. P. Liu and Y. Liu have proved that the operator $T_3=\nabla_{\mathbb{H}^n}(-\Delta_{\mathbb{H}^n}+V)^{-1/2}$ is

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bounded from $H_L^1(\mathbb{H}^n)$ to $L^1(\mathbb{H}^n)$ in [5]. In this paper we will show that the other two operators T_1 and T_2 are also bounded from $H_L^1(\mathbb{H}^n)$ to $L^1(\mathbb{H}^n)$. At the last section, we simply state the results on the stratified Lie group.

In what follows we recall some basic facts for the Heisenberg group $\mathbb{H}^n$ (cf. [11]). The Heisenberg group $\mathbb{H}^n$ is a lie group with the underlying manifold $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$, and the multiplication

$$(x,y,t)(x',y',t')=(x+x',y+y',t+t'+2x'y-2xy').$$

A basis for the Lie algebra of left-invariant vector fields on $\mathbb{H}^n$ is given by

$$X_j=\frac{\partial}{\partial x_j}+2y_j\frac{\partial}{\partial t},Y_{j=}\frac{\partial}{\partial y_j}+2x_j\frac{\partial}{\partial t},T=\frac{\partial}{\partial t},\quad j=1,2,\cdots,n.$$

All non-trivial commutation relations are given by $[X_j,Y_j]=-4T$, $j=1,2,\cdots,n$. Then the sub-Laplacian $\Delta_{\mathbb{H}^n}$ is defined by $\Delta_{\mathbb{H}^n}=\sum_{j=1}^n(X_j^{\,2}+Y_j^{\,2})$ and the gradient operator $\nabla_{\mathbb{H}^n}$ is defined by

$$\nabla_{\mathbb{H}^n}=(X_1,\cdots,X_n,Y_1,\cdots,Y_n).$$

The dilations on $\mathbb{H}^n$ have the form $\delta_\lambda(x,y,t)=(\lambda x,\lambda y,\lambda^2 t),\lambda>0$. The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$. We denote the measure of any measurable set E by $|E|$. Then $|\delta_\lambda E|=\lambda^Q|E|$, where $Q=2n+2$ is called the homogeneous dimension of $\mathbb{H}^n$.

We define a homogeneous norm function on $\mathbb{H}^n$ by

$$|g|=((|x|^2+|y|^2)^2+|t|^2)^{\frac{1}{4}},\quad g=(x,y,t)\in\mathbb{H}^n.$$

This norm satisfies the triangular inequality and leads to a left-invariant distant function $d(g,h)=|g^{-1}h|$. Then the ball of radius r centered at g is given by

$$B(g,r)=\{h\in\mathbb{H}^n:|g^{-1}h|<r\}.$$

The ball $B(g,r)$ is the left translation by g of $B(0,r)$ and we have $|B(g,r)|=\alpha_1 r^Q$, where $\alpha_1=|B(0,1)|$, but it is not important for us.

A nonnegative locally L^q integrable function V on $\mathbb{H}^n$ is said to belong to B_q $(1<q<\infty)$ if there exists $C>0$ such that the reverse Hölder inequality

$$\Big(\frac{1}{|B|}\int_B V(g)^q dg\Big)^{\frac{1}{q}}\le\frac{C}{|B|}\int_B V(g)dg$$

holds for every ball B in $\mathbb{H}^n$.

It is obvious that $B_{q_2}\subset B_{q_1}$ where $q_2>q_1$. From [3] we know that the B_q class has a property of "self improvement"; that is, if $V\in B_q$, then $V\in B_{q+\varepsilon}$ for some $\varepsilon>0$.

Assume that $V\in B_{q_1}$ for some $q_1>Q/2$. The definition of the auxiliary function $m(g,V)$ is given as follows.

Definition 1.1. For $g\in\mathbb{H}^n$, the function $m(g,V)$ is defined by

$$\rho(g)=\frac{1}{m(g,V)}=\sup_{r>0}\Big\{r:\frac{1}{r^{Q-2}}\int_{B(g,r)}V(h)dh\le 1\Big\}.$$

In order to obtain the estimates of T_1 and T_2 on Hardy spaces, we also need to recall the Hardy space associated with the Schrödinger operator L on the Heisenberg group which had been studied in [5] and [12]. The maximal function associated with $\{T_s^L:s>0\}$ is defined by $M^Lf(g)=\sup_{s>0}\big|T_s^Lf(g)\big|$, where $\{T_s^L:s>0\}=\{e^{-sL}:s>0\}$ is the semigroup generated by the Schrödinger operator L. The Hardy space $H_L^1(\mathbb{H}^n)$ is defined as follows.

Definition 1.2. We say that $f\in L^1(\mathbb{H}^n)$ is an element of $H_L^1(\mathbb{H}^n)$ if the maximal function M^Lf belongs to $L^1(\mathbb{H}^n)$. The quasi-norm of f is defined by $\|f\|_{H_L^1(\mathbb{H}^n)}=\|M^Lf\|_{L^1(\mathbb{H}^n)}$.

Definition 1.3. Let $1<q\le\infty$. A function $a\in L^q(\mathbb{H}^n)$ is called an $H_L^{1,q}$-atom if $r\le\rho(g_0)$ and the following conditions hold:

(i) $\operatorname{supp} a\subset B(g_0,r)$, $r>0$,

(ii) $\|a\|_{L^q(\mathbb{H}^n)}\le|B(g_0,r)|^{\frac{1}{q}-1}$,

(iii) if $r<\frac{\rho(g_0)}{4}$, then $\int_{B(g_0,r)}a(g)dg=0$.

It follows from (i) and (ii) in Definition 1.3 that a $H_L^{1,\infty}$ atom is also a $H_L^{1,q}$ atom for $1\le q<\infty$. We have the following atomic characterization by the results in [5] and [12].

Proposition 1.1. Let $1<q\le\infty$ and $f\in L^1(\mathbb{H}^n)$. Then $f\in H_L^1(\mathbb{H}^n)$ if and only if f can be written as $f=\sum_j\lambda_ja_j$, where a_j are $H_L^{1,q}$-atoms,

$$\sum_j|\lambda_j|<\infty,$$

and the sum converges in the $H_L^1(\mathbb{H}^n)$ quasi-norm. Moreover,

$$\|f\|_{H_L^1(\mathbb{H}^n)}\sim\inf\Big\{\sum_j|\lambda_j|\Big\},$$

where the infimum is taken over all atomic decompositions of f into $H_L^{1,q}$-atoms.

The atomic decompositions of $H_L^1(\mathbb{H}^n)$ imply that the space $H_L^1(\mathbb{H}^n)$ is larger than the classical Hardy space $H^1(\mathbb{H}^n)$. Specifically, the Hardy space $H_L^1(\mathbb{H}^n)$ is the local Hardy space $H^1(\mathbb{H}^n)$ if the potential V is a positive constant (cf. [5]).

Now we are in a position to give the main results.

Theorem 1.1. *Suppose $V\in B_{q_1}$, $q_1>Q/2$. Then the operator $T_1=V(-\Delta_{\mathbb{H}^n}+V)^{-1}$ is a bounded linear operator from $H_L^1(\mathbb{H}^n)$ to $L^1(\mathbb{H}^n)$. That is, there exists a positive constant $C>0$ such that*

$$\|T_1f\|_{L^1(\mathbb{H}^n)}\le C\|f\|_{H_L^1(\mathbb{H}^n)}.$$

Theorem 1.2. *Suppose $V\in B_{q_1}$, $q_1>Q/2$. Then the operator $T_2=V^{1/2}(-\Delta_{\mathbb{H}^n}+V)^{-1/2}$ is bounded from $H^1_L(\mathbb{H}^n)$ to $L^1(\mathbb{H}^n)$. That is, there exists a positive constant $C>0$ such that*

$$\|T_2f\|_{L^1(\mathbb{H}^n)}\le C\|f\|_{H^1_L(\mathbb{H}^n)}.$$

Remark 1.1. It is natural to ask whether the operators T_1 and T_2 are bounded from $H^1_L(\mathbb{H}^n)$ into $H^1_L(\mathbb{H}^n)$, even from $H^p_L(\mathbb{H}^n)$ into $H^p_L(\mathbb{H}^n)$ with suitable $p<1$? We think these problems are true. But their proofs depend on the molecular characterization of $H^p_L(\mathbb{H}^n)$. We will investigate the topic in our another paper.

2 The auxiliary function $m(g,V)$

In this section, we will recall some related lemmas about the auxiliary function. Refer to [3] for the proofs. We assume that the potential $V(g)$ is nonnegative and belongs to B_{q_1} for $q_1\ge Q/2$.

Lemma 2.1. *There exists a constant $C>0$ such that, for $0<r<R<\infty$,*

$$\frac{1}{r^{Q-2}}\int_{B(g,r)}V(h)dh\le C\Big(\frac{R}{r}\Big)^{\frac{Q}{q_1}-2}\frac{1}{R^{Q-2}}\int_{B(g,R)}V(h)dh.$$

Lemma 2.2.

$$\frac{1}{r^{Q-2}}\int_{B(g,r)}V(h)dh\sim 1$$

holds if and only if $r\sim\rho(g)$.

Lemma 2.3. *There exist $C>0$ and $l_0>0$ such that*

$$\frac{1}{C}(1+m(g,V)|g^{-1}h|)^{-l_0}\le\frac{m(g,V)}{m(h,V)}\le C(1+|g^{-1}h|m(g,V))^{\frac{l_0}{l_0+1}}.$$

In particular, $\rho(g)\sim\rho(h)$ if $|g^{-1}h|<C\rho(g)$.

Lemma 2.4. *There exist $C>0$ and $l_1>0$ such that*

$$\int_{B(g,R)}\frac{V(h)}{|g^{-1}h|^{Q-2}}dh\le\frac{C}{R^{Q-2}}\int_{B(g,R)}V(h)dh\le C(1+Rm(g,V)g^{-1}h)^{l_1}.$$

3 Estimates of fundamental solution for the Schrödinger operator

In this section we recall some estimates of fundamental solution of the operator $-\Delta_{\mathbb{H}^n}+V+\lambda$ and estimates of the kernels of Riesz transforms. Let $\Gamma(g,h,\lambda)$ be the fundamental solution of the operator $-\Delta_{\mathbb{H}^n}+V+\lambda$, where $\lambda\in[0,\infty)$. Obviously, $\Gamma(g,h,\lambda)=\gamma(h,g,\lambda)$.

The proofs of the following Lemmas have been given in [3].

Lemma 3.1. *Suppose $V\in B_{q_1}$, $q_1>Q/2$. For any integer $N>0$ there exists $C_N>0$ such that for $g\neq h$, we have*

$$|\Gamma(g,h,\lambda)|\le \frac{C_N}{\{1+|g^{-1}h||\lambda|^{1/2}\}^N\{1+|g^{-1}h|\rho(g)^{-1}\}^N}\frac{1}{|g^{-1}h|^{Q-2}}.$$

The operator $T_1=V(-\Delta_{\mathbb{H}^n}+V)^{-1}$ is defined by

$$T_1f(g)=\int_{\mathbb{H}^n}K_1(g,h)f(h)dh,$$

where $K_1(g,h)=V(g)\Gamma(g,h)$ and $\Gamma(g,h)=\Gamma(g,h,0)$. By functional calculus, the operator

$$T_2=V^{\frac{1}{2}}(-\Delta_{\mathbb{H}^n}+V)^{-\frac{1}{2}}$$

is defined by

$$T_2f(g)=\int_{\mathbb{H}^n}K_2(g,h)f(h)dh,$$

where

$$K_2(g,h)=\frac{1}{\pi}\int_0^\infty\lambda^{-\frac{1}{2}}\Gamma(g,h,\lambda)d\lambda V(g)^{1/2}.$$

The proofs of the following lemmas can be found from Lemma 3 and Lemma 4 in [4].

Lemma 3.2. *Suppose $V\in B_{q_1}$, $q_1>Q/2$. For any integer $N>0$ there exists $C_N>0$ such that*

$$|K_1(g,h)|\le\frac{C_N}{\{1+|g^{-1}h|\rho(g)^{-1}\}^N}\frac{V(g)}{|g^{-1}h|^{Q-2}}$$

and

$$|K_1(g,h\xi)-K_1(g,h)|\le\frac{C_N}{\{1+|g^{-1}h|\rho(g)^{-1}\}^N}\frac{|\xi|^\delta}{|g^{-1}h|^{Q-2+\delta}}V(g)$$

for any $g,h\in\mathbb{H}^n$, $|\xi|\le\frac{|g^{-1}h|}{2}$ and some $\delta>0$.

Lemma 3.3. *Suppose $V\in B_{q_1}$, $q>Q/2$. For any integer $N>0$ there exists $C_N>0$ such that*

$$|K_2(g,h)|\le\frac{C_N}{\{1+|g^{-1}h|\rho(g)^{-1}\}^N}\frac{V(g)^{1/2}}{|g^{-1}h|^{Q-1}}$$

and

$$|K_2(g,h\xi)-K_2(g,h)|\le\frac{C_N}{\{1+|g^{-1}h|\rho(g)^{-1}\}^N}\frac{|\xi|^\delta}{|g^{-1}h|^{Q-1+\delta}}V(g)^{1/2}$$

for any $g,h\in\mathbb{H}^n$, $|\xi|\le\frac{|g^{-1}h|}{2}$ and some $\delta>0$.

4 Proofs of main results

The aim of this section is to prove the Hardy type estimates for Riesz transforms T_1 and T_2 on the Heisenberg group $\mathbb{H}^n$.

The following propositions prove the $L^p(\mathbb{H}^n)$ boundedness of Riesz transforms associated with the Schrödinger operator $L=-\Delta_{\mathbb{H}^n}+V$. The proofs have been given in [3].

Proposition 4.1. Suppose $V\in B_{q_1}$, $Q/2\le q_1<Q$, then for $1<p\le q_1$,

$$\|V(-\Delta_{\mathbb{H}^n}+V)^{-1}f\|_{L^p(\mathbb{H}^n)}\le C_p\|f\|_{L^p(\mathbb{H}^n)},$$

where the constant $C_p>0$ doesn't depend on f.

Proposition 4.2. Suppose $V\in B_{q_1}$, $Q/2\le q_1<Q$, then for $1<p\le 2q_1$,

$$\|V^{1/2}(-\Delta_{\mathbb{H}^n}+V)^{-1/2}f\|_{L^p(\mathbb{H}^n)}\le C_p\|f\|_{L^p(\mathbb{H}^n)},$$

where the constant $C_p>0$ doesn't depend on f.

We can arrive at the proof of Theorem 5.1 by the following Lemma.

Lemma 4.1. *Let $q_1>Q/2$. There exists q with $1<q<q_1$ such that*

$$\|T_1a\|_{L^1(\mathbb{H}^n)}\le C$$

for any $H_L^{1,q}$-atom a, where the constant $C>0$ doesn't depend on a.

Proof. Assume that $\operatorname{supp}a\subseteq B(g_0,r)$. We divided into two cases for the proof of the lemma: $r\ge\frac{\rho(g_0)}{4}$ and $r<\frac{\rho(g_0)}{4}$.

Case 1: we consider $r\ge\frac{\rho(g_0)}{4}$. Let $B^*=B(g_0,2r)$, $B^\#=B(g_0,2\rho(g_0))$. Then

$$\|T_1a\|_{L^1(\mathbb{H}^n)}\le\|\chi_{B^*}T_1a\|_{L^1(\mathbb{H}^n)}+\|\chi_{B^{*c}}T_1a\|_{L^1(\mathbb{H}^n)}:=I_1+I_2.$$

According to Proposition 4.1, T_1 is bounded from $L^q(\mathbb{H}^n)$ into $L^q(\mathbb{H}^n)$, thus via the Hölder inequality we get

$$\begin{aligned}I_1&=\Big(\int_{B^*}|T_1a(g)|\Big)\le\Big(\int_{B^*}1dg\Big)^{1-\frac{1}{q}}\Big(\int_{B^*}|T_1a(g)|^qdg\Big)^{\frac{1}{q}}\\&\le C|B|^{1-\frac{1}{q}}\|a\|_{L^q(\mathbb{H}^n)}\le C|B|^{1-\frac{1}{q}}|B|^{\frac{1}{q}-1}=C.\end{aligned}$$

For I_2, using the Minkowski inequality, Lemma 2.3 and Lemma 2.4, noting that $|g^{-1}h|\sim|g^{-1}g_0|$, we have

$$\begin{aligned}
I_2\le&\int_B|a(h)|dh\Big(\int_{B^{*c}}|K_1(g,h)|dg\Big)\\
\le&C_N\int_B|a(h)|dh\Big(\int_{B^{*c}}\frac{V(g)dg}{|g^{-1}h|^{Q-2}(1+|g^{-1}h|\rho(g)^{-1})^N}\Big)\\
\le&C_N\int_B|a(h)|dh\Big(\int_{B^{*c}}\frac{V(g)dg}{|g^{-1}g_0|^{Q-2}(1+|g^{-1}g_0|\rho(g_0)^{-1})^{\frac{N}{l_0+1}}}\Big)\\
\le&C_N\int_B|a(h)|dh\Big(\sum_{j=1}^{\infty}\int_{2^jr<|g^{-1}g_0|\le 2^{j+1}r}\frac{V(g)dg}{(2^jr)^{Q-2}(1+2^j)^{\frac{N}{l_0+1}}}\Big)\\
\le&C_N\int_B|a(h)|dh\Big(\sum_{j=1}^{\infty}\frac{1}{(1+2^j)^{\frac{N}{l_0+1}}}\frac{1}{(2^jr)^{Q-2}}\int_{|g^{-1}g_0|\le 2^{j+1}r}V(g)dg\Big)\\
\le&C_N\int_B|a(h)|dh\Big(\sum_{j=1}^{\infty}\frac{1}{(1+2^j)^{\frac{N}{l_0+1}-l_1}}\Big)\\
\le&C\Big(\int_B|a(h)|^qdh\Big)^{1/q}|B|^{1-1/q}=C,
\end{aligned}$$

where we choose N sufficiently large and use the assumption

$$\frac{\rho(g_0)}{4}\le r\le\rho(g_0).$$

Case 2: we consider $r<\frac{\rho(g_0)}{4}$. At this time, $B^*\subseteq B^\#$ and the atom a is a classical atom. We give the decomposition of the operator T_1 as follows:

$$\begin{aligned}
T_1a(g)=&\int_{\mathbb{H}^n}K_1(g,h)a(h)dh\\
=&\chi_{B^{\#c}}(g)\int_{\mathbb{H}^n}K_1(g,h)a(h)dh+\chi_{B^\#\setminus B^*}(g)\int_{\mathbb{H}^n}[K_1(g,h)-K_1(g,g_0)]a(h)dh\\
&+\chi_{B^*}(g)\int_{\mathbb{H}^n}K_1(g,h)a(h)dh\\
:=&J_1+J_2+J_3,
\end{aligned}$$

then

$$\|T_1a\|_{L^1(\mathbb{H}^n)}\le\|J_1\|_{L^1(\mathbb{H}^n)}+\|J_2\|_{L^1(\mathbb{H}^n)}+\|J_3\|_{L^1(\mathbb{H}^n)}.$$

Obviously, similar to the proof of Case 1, it is easy to get

$$\|J_1\|_{L^1(\mathbb{H}^n)}+\|J_3\|_{L^1(\mathbb{H}^n)}\le C.$$

For J_2. Using Lemma 3.2 and Lemma 2.3, we can get

$$
\begin{aligned}
\|J_2\|_{L^1(\mathbb{H}^n)} \leq & \int_B |a(h)|dh\Big(\int_{B^\# \setminus B^*} |K_1(g,h)-K_1(g,g_0)|dg\Big)\\
\leq & C_N \int_B |a(h)|dh\Big(\int_{B^\# \setminus B^*} \frac{|h^{-1}g_0|^\delta V(g)dg}{(1+|g^{-1}g_0|\rho(g_0)^{-1})^N |g^{-1}g_0|^{Q-2+\delta}}\Big)\\
\leq & C_N \int_B |a(h)|dh(\int_{B^\# \setminus B^*} \frac{|h^{-1}g_0|^\delta V(g)dg}{(1+|g^{-1}g_0|\rho(g_0)^{-1})^{\frac{N}{l_0+1}} |g^{-1}g_0|^{Q-2+\delta}})\\
\leq & C_N \int_B |a(h)|dh\Big(\sum_{j=1}^\infty \int_{2^j r<|g^{-1}g_0|\leq 2^{j+1}r} \frac{r^\delta V(g)dg}{(1+2^j r\rho(g_0)^{-1})^{\frac{N}{l_0+1}} (2^j r)^{Q-2+\delta}}\Big)\\
\leq & C_N \int_B |a(h)|dh\Big(\sum_{j=1}^\infty 2^{-\delta j} \frac{1}{(1+2^j r\rho(g_0)^{-1})^{\frac{N}{l_0+1}}} \frac{1}{(2^j r)^{Q-2}} \int_{|g^{-1}g_0|\leq 2^{j+1}r} V(g)dg\Big)\\
\leq & C_N \int_B |a(h)|dh\Big(\sum_{j=1}^\infty 2^{-\delta j} \frac{1}{(1+2^j r\rho(g_0)^{-1})^{\frac{N}{l_0+1}-l_2}}\Big)\\
\leq & C_N \int_B |a(h)|dh\Big(\sum_{j=1}^\infty 2^{-\delta j}\Big) \leq C,
\end{aligned}
$$

where we choose N sufficiently large. Thus Lemma 4.1 is proved. □

We also arrive at the proof of Theorem 5.2 by the following Lemma.

Lemma 4.2. *Let $q_1 > \frac{Q}{2}$. There exists q with $1<q<2q_1$ such that*

$$\|T_2 a\|_{L^1(\mathbb{H}^n)} \leq C$$

for any $H_L^{1,q}$-atom a, where the constant $C>0$ doesn't depend on a.

Proof. Assume that $\mathrm{supp} a \subseteq B(g_0,r)$. We divided into two cases for the proof of the lemma: $r \geq \frac{\rho(g_0)}{4}$ and $r < \frac{\rho(g_0)}{4}$.

Case 1: we consider $r \geq \frac{\rho(g_0)}{4}$. Let $B^* = B(g_0,2r)$, $B^\# = B(g_0,2\rho(g_0))$. Then

$$\|T_2 a\|_{L^1(\mathbb{H}^n)} \leq \|\chi_{B^*} T_2 a\|_{L^1(\mathbb{H}^n)} + \|\chi_{B^{*c}} T_2 a\|_{L^1(\mathbb{H}^n)} := \tilde{I}_1 + \tilde{I}_2.$$

We choose appropriate $q>1$ such that $1<q<2q_1$. Then according to Proposition 4.2, T_2 is bounded from $L^q(\mathbb{H}^n)$ to $L^q(\mathbb{H}^n)$. So similar to the proof of Case 1 in Lemma 4.1, it is easy to see that $\tilde{I}_1 \leq C$.

For $\tilde{I}_2$, using the Minkowski inequality, Lemma 2.3 and Lemma 2.4, noting that $|g^{-1}h|\sim|g^{-1}g_0|$, we have

$$\begin{aligned}
\tilde{I}_2 \le& \int_B |a(h)|dh\Big(\int_{B^{*c}}|K_2(g,h)|dg\Big)\\
\le& C_N\int_B|a(h)|dh\Big(\int_{B^{*c}}\frac{V(g)^{1/2}dg}{|g^{-1}h|^{Q-1}(1+|g^{-1}h|\rho(g)^{-1})^N}\Big)\\
\le& C_N\int_B|a(h)|dh\Big(\int_{B^{*c}}\frac{V(g)^{1/2}dg}{|g^{-1}g_0|^{Q-1}(1+|g^{-1}g_0|\rho(g_0)^{-1})^{\frac{N}{l_0+1}}}\Big)\\
\le& C_N\int_B|a(h)|dh\Big(\sum_{j=1}^{\infty}\int_{2^jr<|g^{-1}g_0|\le 2^{j+1}r}\frac{V(g)^{1/2}dg}{(2^jr)^{Q-1}(1+2^j)^{\frac{N}{l_0+1}}}\Big)\\
\le& C_N\int_B|a(h)|dh\Big(\sum_{j=1}^{\infty}\frac{1}{(1+2^j)^{\frac{N}{l_0+1}}}\frac{1}{(2^jr)^{Q-1}}\int_{|g^{-1}g_0|\le 2^{j+1}r}V(g)^{1/2}dg\Big)\\
\le& C_N\int_B|a(h)|dh\Big(\sum_{j=1}^{\infty}\frac{1}{(1+2^j)^{\frac{N}{l_0+1}}}\frac{1}{(2^jr)^{Q-1}}\Big\{\int_{|g^{-1}g_0|\le 2^{j+1}r}V(g)^{q_1}dg\Big\}^{\frac{1}{2q_1}}(2^jr)^{(1-\frac{1}{2q_1})Q}\Big)\\
\le& C_N\int_B|a(h)|dh\Big(\sum_{j=1}^{\infty}\frac{1}{(1+2^j)^{\frac{N}{l_0+1}}}\frac{1}{(2^jr)^{-1}}\Big\{\frac{1}{(2^jr)^Q}\int_{|g^{-1}g_0|\le 2^{j+1}r}V(g)^{q_1}dg\Big\}^{\frac{1}{2q_1}}\Big)\\
\le& C_N\int_B|a(h)|dh\Big(\sum_{j=1}^{\infty}\frac{1}{(1+2^j)^{\frac{N}{l_0+1}}}\frac{1}{(2^jr)^{-1}}\Big\{\frac{1}{(2^jr)^Q}\int_{|g^{-1}g_0|\le 2^{j+1}r}V(g)dg\Big\}^{\frac{1}{2}}\Big)\\
\le& C_N\int_B|a(h)|dh\Big(\sum_{j=1}^{\infty}\frac{1}{(1+2^j)^{\frac{N}{l_0+1}}}\Big\{\frac{1}{(2^jr)^{Q-2}}\int_{|g^{-1}g_0|\le 2^{j+1}r}V(g)dg\Big\}^{\frac{1}{2}}\Big)\\
\le& C_N\int_B|a(h)|dh\Big(\sum_{j=1}^{\infty}\frac{1}{(1+2^j)^{\frac{N}{l_0+1}-\frac{l_1}{2}}}\Big)\\
\le& C\Big(\int_B|a(h)|^qdh\Big)^{1/q}|B|^{1-1/q}\\
=& C,
\end{aligned}$$

where we choose N sufficiently large and use the assumption $\frac{\rho(g_0)}{4}\le r\le\rho(g_0)$.

Case 2: we consider $r<\frac{\rho(g_0)}{4}$. At this time, $B^*\subseteq B^\#$ and the atom a is a classical atom. We give the decomposition of the operator T_2 as follows:

$$\begin{aligned}
T_2a(g)=&\int_{\mathbb{H}^n}K_2(g,h)a(h)dh\\
=&\chi_{B^{\#c}}(g)\int_{\mathbb{H}^n}K_2(g,h)a(h)dh+\chi_{B^\#\setminus B^*}(g)\int_{\mathbb{H}^n}[K_2(g,h)-K_2(g,g_0)]a(h)dh\\
&+\chi_{B^*}(g)\int_{\mathbb{H}^n}K_2(g,h)a(h)dh
\end{aligned}$$

$$:=\tilde{J}_1+\tilde{J}_2+\tilde{J}_3,$$

then

$$\|T_2a\|_{L^1(\mathbb{H}^n)}\le\|\tilde{J}_1\|_{L^1(\mathbb{H}^n)}+\|\tilde{J}_2\|_{L^1(\mathbb{H}^n)}+\|\tilde{J}_3\|_{L^1(\mathbb{H}^n)}.$$

Obviously, similar to the proof of Case 1 in the proof of this lemma, we can get

$$\|\tilde{J}_1\|_{L^1(\mathbb{H}^n)}+\|\tilde{J}_3\|_{L^1(\mathbb{H}^n)}\le C.$$

For $\tilde{J}_2$, using Lemma 3.3 and Lemma 2.3, we have

$$\begin{aligned}
\|\tilde{J}_2\|_{L^1(\mathbb{H}^n)}\le&\int_B|a(h)|dh\Big(\int_{B^\#\backslash B^*}|K_2(g,h)-K_2(g,g_0)|dg\Big)\\
\le&C_N\int_B|a(h)|dh\Big(\int_{B^\#\backslash B^*}\frac{|h^{-1}g_0|^\delta V(g)^{1/2}dg}{(1+|g^{-1}g_0|\rho(g_0)^{-1})^N|g^{-1}g_0|^{Q-1+\delta}}\Big)\\
\le&C_N\int_B|a(h)|dh\Big(\int_{B^\#\backslash B^*}\frac{|h^{-1}g_0|^\delta V(g)^{1/2}dg}{(1+|g^{-1}g_0|\rho(g_0)^{-1})^{\frac{N}{l_0+1}}|g^{-1}g_0|^{Q-1+\delta}}\Big)\\
\le&C_N\int_B|a(h)|dh\Big(\sum_{j=1}^\infty\int_{2^jr<|g^{-1}g_0|\le2^{j+1}r}\frac{r^\delta V(g)^{1/2}dg}{(1+2^jr\rho(g_0)^{-1})^{\frac{N}{l_0+1}}(2^jr)^{Q-1+\delta}}\Big)\\
\le&C_N\int_B|a(h)|dh\Big(\sum_{j=1}^\infty2^{-\delta j}\frac{1}{(1+2^jr\rho(g_0)^{-1})^{\frac{N}{l_0+1}}}\frac{1}{(2^jr)^{Q-1}}\int_{|g^{-1}g_0|\le2^{j+1}r}V(g)^{1/2}dg\Big)\\
\le&C_N\int_B|a(h)|dh\Big(\sum_{j=1}^\infty2^{-\delta j}\frac{1}{(2^jr)^{Q-1}}\frac{1}{(1+2^jr\rho(g_0)^{-1})^{\frac{N}{l_0+1}}}\Big\{\int_{|g^{-1}g_0|\le2^{j+1}r}V(g)^{q_1}dg\Big\}^{\frac{1}{2q_1}}(2^jr)^{(1-\frac{1}{2q_1})Q}\Big)\\
\le&C_N\int_B|a(h)|dh\Big(\sum_{j=1}^\infty2^{-\delta j}\frac{1}{(1+2^jr\rho(g_0)^{-1})^{\frac{N}{l_0+1}}}\frac{1}{(2^jr)^{-1}}\Big\{\int_{|g^{-1}g_0|\le2^{j+1}r}V(g)^{q_1}dg\Big\}^{\frac{1}{2q_1}}\Big)\\
\le&C_N\int_B|a(h)|dh\Big(\sum_{j=1}^\infty2^{-\delta j}\frac{1}{(1+2^jr\rho(g_0)^{-1})^{\frac{N}{l_0+1}}}\Big\{\frac{1}{(2^jr)^{Q-2}}\int_{|g^{-1}g_0|\le2^{j+1}r}V(g)dg\Big\}^{\frac{1}{2}}\Big)\\
\le&C_N\int_B|a(h)|dh\Big(\sum_{j=1}^\infty2^{-\delta j}\frac{1}{(1+2^jr\rho(g_0)^{-1})^{\frac{N}{l_0+1}-\frac{l_1}{2}}}\Big)\\
\le&C_N\int_B|a(h)|dh\Big(\sum_{j=1}^\infty2^{-\delta j}\Big)\\
\le&C,
\end{aligned}$$

where we choose N sufficiently large. Thus this completes the proof of Lemma 4.2. □

5 Results for stratified groups

In this section, we state results for stratified groups. We consistently use the same notations and terminologies as those in Folland and Stein's book [1].

A Lie group G is called stratified if it is nilpotent, connected and simple connected, and its Lie algebra $\mathfrak{g}$ admits a vector space decomposition $\mathfrak{g}=V_1\bigoplus\cdots\bigoplus V_m$ such that $[V_1,V_k]=V_{k+1}$ for $1\le k<m$ and $[V_1,V_m]=0$. If G is stratified, its Lie algebra admits a family of dilations, namely,

$$\delta_r(X_1+X_2+\cdots+X_m)=rX_1+r^2X^2+\cdots+r^mX^m(X_j\in V_j,j\in\{1,\cdots,m\}).$$

Assume that G is a Lie group with underlying manifold $\mathbb{R}^n$ for some positive integer n. G inherits dilations from $\mathfrak{g}$: if $x\in G$ and $r>0$, we write

$$\delta_r x=(r^{d_1}x_1,\cdots,r^{d_n}x_n),$$

where $1\le d_1\le\cdots\le d_n$. The map $x\to\delta_r x$ is an automorphism of G. The left (or right) Haar measure on G is simply $dx_1\cdots dx_n$, which is the Lebesgue measure on $\mathfrak{g}$. For any measurable set $E\subseteq G$, denote by $|E|$ the measure of E. The inverse of any $x\in G$ is simply $x^{-1}=-x$. The group law has the following form

$$xy=(p_1(x,y),\cdots,p_n(x,y)) \tag{5.1}$$

for some polynomials $p_1,\cdots,p_n$ in $x_1,\cdots,x_n,y_1,\cdots,y_n$.

The number $Q=\sum_{j=1}^m j(\dim V_j)$ is called the homogeneous dimension of G. We fix a homogeneous norm function $|\cdot|$ on G, which is smooth away from e, where e is the unit element of G. Thus, $|\delta_r x|=r|x|$ for all $x\in G,r>0,|x^{-1}|=|x|$ for all $x\in G$, and $|x|>0$ if $x\neq 0$. The homogeneous norm induces a quasi-metric d which is defined by $d(x,y):=|x^{-1}y|$. In particularly, $d(e,x)=|x|$ and $d(x,y)=d(e,x^{-1}y)$. The ball of radius r centered at x is written by

$$B(x,r)=\{y\in G|d(x,y)<r\}.$$

The measure of $B(x,r)$ is

$$|B(x,r)|=br^Q,$$

where b is a constant.

Let $X=\{X_1,\cdots,X_l\}$ be a basis for V_1 (viewed as left-invariant vector fields on G). It follows from [1] that X_j, $j=1,2,\cdots,l$, are skew adjoint, that is, $X_j^*=-X_j$. Let $\Delta_G=\sum_{i=1}^l X_i^2$ be the sub-Laplacian on G. It follows from the definition of the stratified Lie group that the Heisenberg group is a special stratified Lie group.

The corresponding results on the stratified Lie group are given as follows:

Theorem 5.1. *Suppose $V\in B_{q_1}$, $q_1>Q/2$. Then the operator $T_1=V(-\Delta_G+V)^{-1}$ is a bounded linear operator from $H_L^1(G)$ to $L^1(G)$. That is, there exists a positive constant $C>0$ such that*

$$\|T_1f\|_{L^1(G)}\le C\|f\|_{H_L^1(G)}.$$

Theorem 5.2. *Suppose $V\in B_{q_1}$, $q_1>Q/2$. Then the operator $T_2=V^{1/2}(-\Delta_G+V)^{-1/2}$ is bounded from $H_L^1(G)$ to $L^1(G)$. That is, there exists a positive constant $C>0$ such that*

$$\|T_2f\|_{L^1(G)}\le C\|f\|_{H_L^1(G)}.$$

References

[1] G. B. Folland and E. M. Stein, Hardy Spaces on Homogeneous Groups, Princeton University Press, Princeton, 1982.

[2] R. M. Gong, J. Li and L. Song, Besov and Hardy spaces associated with the Schrödinger operator on the Heisenberg group, J. Geom. Anal., 24 (2014), 144–168.

[3] H. Q. Li, Estimations L^p des opérateurs de Schrödinger sur les groupes nilpotents, J. Func. Anal., 161 (1999), 152–218.

[4] P. T. Li and L. Z. Peng, L^p boundedness of commutator operator associated with Schrödinger operators on Heisenberg group, Acta Math. Sci. Ser. B Engl. Ed., 32 (2012), 568–578.

[5] C. C. Lin, H. Liu and Y. Liu, Hardy spaces associated with Schrödinger operators on the Heisenberg group, preprint available at arXiv: 1106.4960.

[6] C. C. Lin and H. P. Liu, $BMO_L(\mathbb{H}^n)$ spaces and Carleson measures for Schrödinger operators, Adv. Math., 228 (2011), 1631–1688.

[7] Y. Liu, L^p Estimates for Schrödinger operators on the Heisenberg group, J. Korean Math. Soc., 47 (2010), 425–443.

[8] Y. Liu, J. Z. Huang and D. M. Xie, Some estimates of Schrödinger type operators on the Heisenberg group, Arch. Math., 94 (2010), 255–264.

[9] Y. Liu, J. Z. Huang and J. F. Dong, An estimate on the heat kernel of Schrödinger operators with non-negative potentials on nilpotent Lie groups and its applications, Forum Math., 27 (2015), 1773–1798.

[10] Y. Liu and W. J. Xie, Riesz transform related to Schrödinger type operator on the Heisenberg group, Georgian Math. J., 22 (2015), 397–407.

[11] E. M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, Princeton, 1993.

[12] D. Yang and Y. Zhou, Localized Hardy spaces H^1 related to admissible functions on RD-spaces and applications to Schrödinger operators, Trans. Amer. Math. Soc., 363 (2011), 1197–1239.