Analysis in Theory and Applications
Anal. Theory Appl., Vol. **40**, No. 3 (2024), pp. 221-239

DOI: 10.4208/ata.OA-2022-0013
September 2024

Direct, Inverse Theorems of Approximation by Linear Combinations of Weighted Baskakov–Durrmeyer Operators in Orlicz Spaces

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Received 19 April 2022; Accepted (in revised version) 25 March 2024

Abstract. In the paper, the authors introduce the Orlicz space corresponding to the Young function and, by virtue of the equivalent theorem between the modified K-functional and modulus of smoothness, establish the direct, inverse, and equivalent theorems for the linear combinations of Jacobi weighted Baskakov–Durrmeyer operators in the Orlicz spaces.

Key Words: Direct theorem, inverse theorem, equivalent theorem, approximation, K-functional, Orlicz space, Jacobi weight, Baskakov–Durrmeyer operator.

AMS Subject Classifications: 41A17, 41A27, 41A35

1 Preliminaries, motivations, and main results

Throughout this paper, we use C to denote a constant independent of n and x, which may be not necessarily the same in different cases, and use $\mathbb{N}_0$ to denote the set $\{0,1,2,\cdots\} = \{0\}\cup\mathbb{N}$ of all nonnegative integers. In recent years, since the Orlicz spaces are more general than the classical L_p spaces, which are composed of measurable functions $f(x)$ such that $|f(x)|^p$ are integrable, there is growing interest in problems of approximation in Orlicz spaces. For proceeding smoothly, we recall from [16, 19] some definitions and related results. A continuous convex function $\Phi(t)$ on $[0,\infty)$ is called a Young function if

$$\lim_{t\to 0^+}\frac{\Phi(t)}{t}=0 \quad \text{and} \quad \lim_{t\to\infty}\frac{\Phi(t)}{t}=\infty.$$

For a Young function $\Phi(t)$, its complementary Young function is denoted by $\Psi(t)$. It is clear that the convexity of $\Phi(t)$ leads to $\Phi(\alpha t)\le\alpha\Phi(t)$ for $\alpha\in[0,1]$. In particular,

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we have $\Phi(\alpha t) < \alpha\Phi(t)$ for $\alpha \in (0,1)$. A Young function $\Phi(t)$ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if there exist $t_0 \geq 0$ and $C \geq 1$ such that $\Phi(2t) \leq C\Phi(t)$ for $t \geq t_0$. Let $\Phi(t)$ be a Young function. We define the Orlicz class $L_\Phi[0,\infty)$ as the collection of all Lebesgue measurable functions $u(x)$ on $[0,\infty)$ for which

$$\rho(u,\Phi) = \int_0^\infty \Phi(|u(x)|)\mathrm{d}x < \infty$$

and define the Orlicz space $L^*_\Phi[0,\infty)$ as the collection of all Lebesgue measurable functions $u(x)$ on $[0,\infty)$, such that

$$\int_0^\infty \Phi(|\alpha u(x)|)\mathrm{d}x < \infty$$

for some $\alpha > 0$. The Orlicz space is a Banach space under the Luxemburg norm

$$\|u\|_{(\Phi)} = \inf_{\lambda>0}\Big\{\lambda : \rho\Big(\frac{u}{\lambda},\Phi\Big) \leq 1\Big\}.$$

The Orlicz norm, which is equivalent to the Luxemburg norm on $L^*_\Phi[0,\infty)$, is given by

$$\|u\|_\Phi = \sup_{\rho(v,\Psi)\leq 1}\Big|\int_0^\infty u(x)v(x)\mathrm{d}x\Big|$$

and satisfies

$$\|u\|_{(\Phi)} \leq \|u\|_\Phi \leq 2\|u\|_{(\Phi)}. \tag{1.1}$$

If $\Phi(u) = \frac{u^p}{p}$ for $1 < p < \infty$, then the complementary function becomes

$$\Psi(u) = \frac{|u|^q}{q} \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1$$

and then $L^*_\Phi[0,\infty) = L_p[0,\infty)$. So the Orlicz spaces $L^*_\Phi[0,\infty)$ are more general than the classical $L_p[0,\infty)$ spaces which are composed of measurable functions $f(x)$ such that $|f(x)|^p$ are integrable.

Let

$$\begin{aligned} &L^*_{\Phi,w}[0,\infty) \triangleq \{f : wf \in L^*_\Phi[0,\infty)\}, \quad f \in L^*_{\Phi,w}[0,\infty), \quad r \in \mathbb{N}, \\ &w(x) = x^a(1+x)^b \quad \text{for} \quad a,b \in \mathbb{R}, \end{aligned}$$

is the Jacobi weight function. Then the weighted K-functional $K_{r,\varphi}(f,t^r)_{w,\Phi}$, the weighted modified K-functional $\overline{K}_{r,\varphi}(f,t^r)_{w,\Phi}$, and the weighted modulus of smoothness