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The Convexity of Inclusions and Gradient's Concentration for Lamé Systems with Partially Infinite Coefficients

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Abstract. It is interesting to study the stress concentration between two adjacent stiff inclusions in composite materials, which can be modeled by the Lamé system with partially infinite coefficients. To overcome the difficulty from the lack of maximum principle for elliptic systems, we use the energy method and an iteration technique to study the gradient estimates of the solution. We first find a novel phenomenon that the gradient will not blow up any more once these two adjacent inclusions fail to be locally relatively strictly convex, namely, the top and bottom boundaries of the narrow region are partially "flat". In order to further explore the blow-up mechanism of the gradient, we next investigate two adjacent inclusions with relative convexity of order m and finally reveal an underlying relationship between the blow-up rate of the stress and the order of the relative convexity of the subdomains in all dimensions.

Key Words: Gradient estimates, convex inclusions, blowup analysis, linear elasticity, composite materials.

AMS Subject Classifications: 35Q74, 35B65, 74G70

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1 Introduction and main results

The convexity plays a central role in many questions in analysis. The purpose of this paper is mainly to investigate the significant role of the relative convexity between two adjacent inclusions in the blow-up analysis of the stress in high-contrast fiber-reinforced composite materials, where the inclusions are frequently spaced very closely and even touching. This work is motivated by issue of material failure initiation, where it is well known that high concentration phenomenon of mechanical loads in the extreme loads

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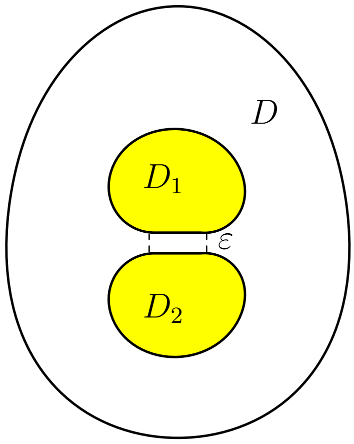

Figure 1: Two adjacent inclusions with partially "flat" boundaries, like an egg with double yolks.

will be amplified by the composite microstructure, for example, the narrow region between two adjacent inclusions. However, whenever the narrow region has certain partially "flat" top and bottom boundaries (see Fig. 1), we prove that the gradient of the solutions to Lamé systems with partially infinite coefficients, is bounded by some positive constant, independent of the distance between the inclusions, rather than blows up as one might expect. In order to further explore the blow-up mechanism of the stress, we next investigate two adjacent inclusions with relative convexity of order m, $m \geq 2$, and finally reveal an underlying relationship between the blow-up rate of the stress and the order of the relative convexity of the subdomains in all dimensions. This shows that the relative convexity between inclusions is critical for the stress concentration phenomenon in composite materials.

For strictly convex inclusions, especially for circular inclusions, there have been many important works on the gradient estimates for solution to a class of divergence form elliptic equations and systems with discontinuous coefficients, arising from the study of composite media. For two adjacent disks in dimension two with ε apart, Keller [28] was the first to use analysis to estimate the effective properties of particle reinforced composites. In [8], Babuška, Andersson, Smith, and Levin numerically analyzed the initiation and growth of damage in composite materials, where the Lamé system is assumed. Bonnetier and Vogelius [16] and Li and Vogelius [36] proved the uniform boundedness of $|\nabla u|$ regardless of ε provided that the coefficients stay away from 0 and ∞. Li and Nirenberg [35] extended the results in [36] to general divergence form second order elliptic systems including systems of linear elasticity.

On the other hand, in order to investigate the high-contrast conductivity problem and establish the relationship between $|\nabla u|$ and the distance ε, Ammari, Kang, and Lim [3] studied two close-to-touching disks whose conductivity degenerate to ∞ or 0, a lower bound on $|\nabla u|$ was constructed there showing blow-up of order $\varepsilon^{-1/2}$ in dimension two.