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Examples of Twice Differentiable Functions with Continuous Laplacian and Bounded Discontinuous Hessian

Yifei Pan[1] and Yu Yan[2,*]

[1] *Department of Mathematical Sciences, Purdue University Fort Wayne, Fort Wayne, Indiana 46805, USA*
[2] *Department of Mathematics and Computer Science, Biola University, La Mirada, California 90639, USA*

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Abstract. We construct examples of twice differentiable functions in $\mathbb{R}^n$ with continuous Laplacian and bounded discontinuous Hessian. The same construction is also applicable to higher order differentiability, the Monge-Ampère equation, and the mean curvature equation for hypersurfaces.

Key Words: Twice differentiable, continuous Laplacian, bounded Hessian.

AMS Subject Classifications: 35Axx

1 Introduction

One of the central themes in elliptic theory is to understand the extent to which the Laplacian controls a function's regularity. By the classical Schauder theory, if $\Delta u \in C^{0,\alpha}$ $(0 < \alpha < 1)$, then $u \in C^{2,\alpha}$, but this is false when $\alpha = 0$. There are well known functions [5] that have continuous Laplacian yet are not C^2; for example,

$$w(x,y) = \begin{cases} (x^2 - y^2)\ln(-\ln(x^2+y^2)), & 0 < x^2+y^2 \leq \dfrac{1}{4}, \\ 0, & (x,y) = (0,0). \end{cases}$$

Precisely, this function is not C^2 because it fails to be twice differentiable at the origin. This phenomenon raises the natural question of whether continuous Laplacian and twice differentiability everywhere would be sufficient to guarantee continuous twice differentiability, i.e., C^2. To answer this question, we constructed [6] a large family of functions

*Corresponding author. *Email addresses:* `pan@pfw.edu` (Y. Pan), `yu.yan@biola.edu` (Y. Yan)

that are twice differentiable everywhere with continuous Laplacian but unbounded Hessian, hence not C^2. In light of the existence of such functions one naturally wonders, is an everywhere twice differentiable function with continuous Laplacian and bounded Hessian necessarily C^2?

If twice differentiability everywhere is not required, then there are already simple examples of functions with continuous Laplacian and bounded Hessian that are twice differentiable except at the origin, such as [5]

$$\phi(x,y)=\begin{cases} (x^2-y^2)\sin(\ln(-\ln(x^2+y^2))), & 0<x^2+y^2\le \dfrac{1}{4}, \\ 0, & (x,y)=(0,0). \end{cases}$$

Our first observation is that any possible counter-examples must be non-radially symmetric due to the following result which will be proved in Section 5.

Proposition 1.1. *Any twice differentiable, radially symmetric function with continuous Laplacian must be C^2.*

A similar situation (of twice differentiability) arose in the study of parabolic equations related to the mean curvature flow. If for each s the level sets

$$M_t=\{x|v(x,t)=s, \text{ where } v:\mathbb{R}^{n+1}\times\mathbb{R}\to\mathbb{R}\}$$

evolve by the mean curvature flow, then v satisfies the level set equation

$$\partial_t v=|\nabla v|\,\mathrm{div}\left(\frac{\nabla v}{|\nabla v|}\right).$$

In general, v is a weak solution in the viscosity sense, so it may not be differentiable. When the initial hypersurface is mean convex, Evans and Spruck [4] showed that $v(x,t)=u(x)-t$, where u is a Lipschitz function and satisfies (in the viscosity sense)

$$-1=|\nabla u|\,\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right).$$

The function u is called the arrival time. Colding and Minicozzi proved in [3] that surprisingly u is twice differentiable everywhere with bounded Hessian, furthermore, it satisfies the equation everywhere in the classical sense. It is intriguing to know whether Δu is continuous without u being C^2. If true, that would provide theoretical examples of twice differentiable functions with continuous Laplacian and bounded Hessian that are not C^2.

In this paper we will show the existence of twice differentiable functions with continuous Laplacian and bounded (but discontinuous) Hessian, thus answering the aforementioned question in the negative. That is, an everywhere twice differentiable function with continuous Laplacian and bounded Hessian is not necessarily C^2.