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On Durrmeyer Type Bernstein-Schurer Operators Defined by (p,q)-Integers

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Abstract. In this paper, we generalize the Durrmeyer-type Bernstein-Schurer operator by applying (p,q)-integers and obtain uniform convergence of the operator. Furthermore, we deal with the approximation problems in terms of the modulus of smoothness and K-functional. Finally, the operator is modified to get better estimation.

Key Words: (p,q)-integers, (p,q)-Durrmeyer-Schurer operator, modulus of smoothness, Lipschitz-class.

AMS Subject Classifications: 41A10, 41A25, 41A36

1 Introduction

In recent years, Mursaleen et al. proposed the (p,q)-Bernstein operator by applying (p,q)-integers [1]. Later, many (p,q)-type operators are studied by some authors, such as modified Bernstein-Schurer operators [2–4], (p,q)-Bernstein-Stancu operators [5], (p,q)-Bleimann-Butzer-Hahn operators [6], (p,q)-Lorentz polynomials [7], (p,q)-Szász-Mirakyan operators [8], (p,q)-Durrmeyer operators [9] and so on. In this paper, we study the (p,q)-Durrmeyer-Schurer operator and it's approximation properties.

Firstly, we recall some notations of (p,q)-integers. Throughout this article p and q satisfy $0 < q < p \leq 1$, for any nonnegative number k, the (p,q)-integer and (p,q)-factorial are defined as

$$[k]_{p,q} = \frac{p^k - q^k}{p-q}, \qquad k = 0, 1, \cdots,$$

$$[k]_{p,q}! = \begin{cases} [k]_{p,q}[k-1]_{p,q}\cdots[1]_{p,q}, & k \geq 1, \\ 1, & k = 0. \end{cases}$$

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The (p,q)-binomial coefficients are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \quad 0 \le k \le n,$$

and the (p,q)-binomial expansion is defined as

$$(x+y)^n_{p,q} = (x+y)(px+qy)(p^2x+q^2y)\cdots(p^{n-1}x+q^{n-1}y).$$

For $p=1$, (p,q)-integers in above equalities turn out to be q-integers. Also we can convert (p,q)-calculus into q-calculus by $[k]_{p,q} = p^{k-1}[k]_{q/p}$. More details about (p,q)-calculus can see in [10–12].

In [13], Barbosu proposed the Durrmeyer-Schurer operator $D_{n+s} : C[0,1+s] \to C[0,1]$ is defined for any $f \in C[0,1+s]$,

$$D_{n+s}(f;x) = (n+s+1)\sum_{k=0}^{n+s} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt,$$

where $x \in [0,1]$ and

$$p_{n,k}(x) = \binom{n+s}{k} x^k(1-x)^{n+s-k}.$$

In this paper, we generalize the above operator and study the approximation properties by means of modulus of smoothness and K-functional.

2 Construction of operators

In this section, we construct the (p,q)-type Durrmeyer-Schurer operator as

$$D_{n+s}(f;p,q,x) = [n+s+1]_{p,q}p^{-(n+s)^2}\sum_{k=0}^{n+s} b_{n+s,k}(x)\Big(\frac{q}{p}\Big)^{-k}\int_0^1 b_{n+s,k}(qt)f(t)d_{p,q}t,$$

where $x \in [0,1]$ and

$$b_{n+s,k}(x) = p^{\frac{k(k-1)}{2}}\begin{bmatrix} n+s \\ k \end{bmatrix}_{p,q} x^k(1-x)^{n+s-k}_{p,q}.$$

For $0 < q < p \le 1$ and $f \in C[0,1+s]$, the (p,q)-Bernstein-Schurer operator is defined as:

$$B_{n+s}(f;p,q,x) = p^{\frac{1}{(n+s)(n+s-1)/2}}\sum_{k=0}^{n+s} b_{n+s,k}(x)f\Big(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}\Big),$$

where $b_{n+s,k}(x)$ is given in the above equality and the moments of (p,q)-Bernstein-Schurer can be obtain as: