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Toeplitz O-Frames for Operators in Banach Spaces

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Abstract. We define Toeplitz O-frame for operators as a generalization of the notion of O-frame introduced by Reinov [11]. A necessary condition for the existence of a Toeplitz O-frame is given. It has been proved that an O-frame for operators can generate a Toeplitz O-frame from a given Toeplitz matrix but the converse need not be true. Also, a sufficient condition on infinite matrices for the existence of an O-frame is given. Finally, the notion of a strong O-frame is defined and a necessary and sufficient condition for its existence has been obtained.

Key Words: Frames, operators, O-frames.

AMS Subject Classifications: 42C15, 46B28

1 Introduction

Frames were first introduced by Duffin and Schaeffer [4] in the context of nonharmonic Fourier series. Frames are widely used now a days in applied mathematics and engineering. Feichtinger and Grochenig [5] generalized frames to Banach spaces and introduced the notion of atomic decomposition. Grochenig [6] also introduced a more general concept for Banach spaces called Banach frame. For a nice and comprehensive survey of frames and related concepts one may refer to the text books by Christensen [3] and Heil [7] and the survey article of Casazza [1].

Schauder frames for Banach spaces were introduced by Han and Larson [8] as an inner direct summand of Schauder basis. Schauder frames are used to represent an arbitrary element f of a function space E as a series expansion involving a fixed countable set $\{f_k\}$ of elements in that space such that the coefficients of the expansion of f depend in a linear and continuous way on f. Unlike Schauder bases, the expression of an element f in terms of the elements of a Schauder frame $\{f_k\}$, i.e., the reconstruction formula for f, is

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not necessarily unique. Schauder frames were further studied in [2,9,10]. O. Reinov [11] introduced and studied O-frame for an operator as a generalization of Schauder frame.

In this paper, we introduce and study Toeplitz O-frame for an operator in Banach spaces as a generalization of O-frame and gave a necessary condition for the existence of a Toeplitz O-frame for an operator. It is proved that an O-frame for an operator can generate a Toeplitz O-frame from a given Toeplitz matrix but not conversely. Also, a sufficient condition on infinite matrices for the existence of an O-frame for an operator is proved. Furthur, a perturbation type result for a Toeplitz O-frame is given. Finally, the notion of a strong O-frame for an operator is introduced and a relation between triangular O-frame and strong O-frame is obtained.

2 Preliminaries

Through this paper E will denote a separable Banach space and E^* denote the dual space of E.

Definition 2.1. *An infinite matrix $A = (a_{i,j})$ is called a Toeplitz matrix if the following conditions are satisfied:*

1. $\lim\limits_{i\to\infty} \sum_{j=1}^{\infty} a_{i,j} = 1$,
2. $\lim\limits_{i\to\infty} a_{i,j} = 0$, *for all* $j \in \mathbb{N}$,
3. $\sum_{j=1}^{\infty} |a_{i,j}| \leq K$, *for all* $i \in \mathbb{N}$,

where K is some finite positive constant.

Definition 2.2 ([8])**.** *Let E be a Banach space. A pair of sequence $(\{f_k\},\ \{f_k^*\}) \subset E \times E^*$ is called a Schauder frame for E if*

$$f = \sum_{k=1}^{\infty} f_k^*(f) f_k, \quad f \in E.$$

Reinov [11] generalised this definition and defined the notion of an O-frame as follows:

Definition 2.3. *Let E and F be infinite dimensional separable Banach spaces over the scalar field ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Let $(\{f_k^*\}, \{g_k\}) \subset E^* \times F$ and $T \in B(E,F)$. We say that the pair $(\{f_k^*\}, \{g_k\})$ is an O-*frame *for T if*

$$Tf = \sum_{k=1}^{\infty} f_k^*(f) g_k, \quad f \in E, \tag{2.1}$$

where the series in (2.1) *converges in the norm of F.*

Clearly, an O-frame $(\{f_k^*\}, \{g_k\})$ for $T = I$ is a Schauder frame for E.