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Bounds for the Zeros of Polynomials

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Abstract. In this paper, we present certain results on bounds for the zeros of a polynomial. Our results yield some generalizations and refinements of the recently proved results due to Soleiman and Bidkham, Suhail, Rather and Thakur and other classical results also.

Key Words: Polynomial, zeros, Cauchy theorem.

AMS Subject Classifications: 30C10, 30C15

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1 Introduction

The task of determining the zeros of polynomials has been frequently investigated since the time of Gauss and Cauchy and has played an important role in many disciplines (see [3–5]). Problems involving polynomials in general, and location of their zeros in particular, besides being of particular interest, have important applications in many areas of applied mathematics such as Control theory, Signal processing, Communication theory, Coding theory and Cryptography. An accurate estimate of the annulus containing all the zeros of a polynomial can considerably reduce the amount of work needed to find the exact zeros and so there is always a need for better and better estimates for the annulus containing all the zeros of a polynomial. An account of several developments on this topic can be found in the comprehensive book of Marden [3]. Let

$$P(z)=\sum_{j=o}^{n} a_j z^j$$

be a polynomial of degree n. Then concerning a region which contains all the zeros of $P(z)$, we have the following classical result from Cauchy [3].

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Theorem 1.1. *All the zeros of the complex polynomial $P(z) = \sum_{j=o}^{n} a_j z^j$ lie in the disk*

$$|z| \leq 1 + M,$$

where $M = \max_{0\leq j\leq n-1} \left|\frac{a_j}{a_n}\right|$.

In the literature, there exists several improvements and generalizations of the above result (for example see [3–5]). Recently Gulzar, Rather and Thakur [2] obtained a family of circles each of which contains the zeros of a polynomial and provides a refinement of zero bounds obtained by Cauchy, Tôya, Charmicheal and Mason, Williams [3, pp. 122–126] and others. More precisely, they proved the following result.

Theorem 1.2. *All the zeros of polynomial $P(z) = \sum_{j=o}^{n} a_j z^j$ of degree n lie in the disk*

$$|z| \leq (1 + M_p^q)^{\frac{1}{q}},$$

where

$$M_p = \inf_{\lambda\in\mathbb{C}} \left\{ \sum_{j=0}^{n} \left| \frac{\lambda a_j - a_{j-1}}{a_n} \right|^p \right\}^{\frac{1}{p}}, \quad a_{-1} = 0, \quad p > 1, \quad q > 1,$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

From the definition of M_p above, we have

$$M_p \leq \left\{ \sum_{j=0}^{n} \left| \frac{\lambda a_j - a_{j-1}}{a_n} \right|^p \right\}^{\frac{1}{p}} \quad \text{for each } \lambda \in \mathbb{C}.$$

In particular, for $\lambda = 1$, we get the following result from Theorem 1.2 recently proved by Soleiman and Bidkham [6].

Theorem 1.3. *All the zeros of polynomial $P(z) = \sum_{j=o}^{n} a_j z^j$ of degree n lie in the disk*

$$K\big(0,\ (1 + A_{p,n}^q)^{\frac{1}{q}}\big),$$

where

$$A_{p,n} = \left\{ \sum_{j=0}^{n} \left| \frac{a_{n-j} - a_{n-j-1}}{a_n} \right|^p \right\}^{\frac{1}{p}}, \quad a_{-1} = 0, \quad p > 1, \quad q > 1,$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

In this paper, we first prove the following result that sharpens Theorem 1.2 for a particular λ by deriving an improved bound for moduli of all the zeros of polynomial $P(z)$. To be precise, we prove