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Generalization of Certain Hyperbolic Integrals and a Dilogarithm Functional Relation

F. M. S. Lima*

Institute of Physics, University of Brasilia, P.O. Box 04455, 70919-970, Brasilia-DF, Brazil

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Abstract. In a previous work [Indag. Math., 23(1) (2012)], I did employ a hyperbolic version of the Beukers, Calabi, and Kolk change of variables to solve

$$\int_0^1 \int_0^1 \left(1 - x^2 y^2\right)^{-1} dx dy,$$

which yielded exact closed-form expressions for some definite integrals and, from one of them, I proved a two-term dilogarithm identity. Here in this note, I derive closed-form expressions for

$$\int_0^b [\sinh^{-1}(\cosh x) - x] dx, \quad b \geq 0 \quad \text{and} \quad \int_{\alpha/2}^{\beta/2} \ln(\tanh x) dx, \quad b \in \mathbb{R},$$

where $\alpha := \sinh^{-1}(1)$ and $\beta := b + \sinh^{-1}(\cosh b)$. From these general results, I derive a dilogarithm functional relation.

Key Words: Hyperbolic integrals, dilogarithm function, dilogarithm relations.

AMS Subject Classifications: 40C10, 11M06, 33B30

1 Introduction

The beautiful sum formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

first proved by Euler in 1734 and published in 1740 [4], has motivated a search for distinct, modern proofs (see, e.g., [3, 5]). Among these proofs, one finds that by Beukers, Calabi,

*Corresponding author. *Email address:* `fmsl@unb.br` (F. M. S. Lima)

and Kolk (BCK) [2], in which the non-trivial change of variables $x = \sin u/\cos v$ and $y = \sin v/\cos u$ is applied in order to show that

$$\int_0^1 \int_0^1 (1-x^2y^2)^{-1} dxdy = \iint_T dudv,$$

T being the triangular domain $\{(u,v) : u \geq 0,\, v \geq 0,\, u+v \leq \pi/2\}$, whose area equals $\pi^2/8$. This implies that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8},$$

which is equivalent to Euler's result, above.

In a recent work [7], I have applied a hyperbolic version of the BCK change of variables, namely

$$x = \frac{\sinh u}{\cosh v} \qquad \text{and} \qquad y = \frac{\sinh v}{\cosh u}, \tag{1.1}$$

to solve that unit-square integral, which yielded exact closed-form expressions for some hyperbolic integrals, in particular (see Theorems 1 and 3 of [7])

$$\int_0^{\infty} \left[\sinh^{-1}(\cosh u) - u\right] du = \frac{\pi^2}{16}, \tag{1.2a}$$

$$\int_{\alpha/2}^{\infty} \ln(\tanh z) dz = \frac{\alpha^2}{4} - \frac{\pi^2}{16}, \tag{1.2b}$$

where $\alpha := \sinh^{-1}(1) = \ln(\sqrt{2}+1)$. Moreover, on substituting $t = \tanh z$ in the latter integral and expanding the resulting integrand in partial fractions I could show that Eq. (1.2b) corresponds to the following two-term dilogarithm identity (see Theorem 4 of [7]):

$$\mathrm{Li}_2\left(\sqrt{2}-1\right) + \mathrm{Li}_2\left(1-\frac{1}{\sqrt{2}}\right) = \frac{\pi^2 - \ln^2 2}{8} - \frac{\alpha^2}{2}, \tag{1.3}$$

where

$$\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} z^n/n^2,$$

$|z| \leq 1$, is the dilogarithm function [6].†

Here in this note, I generalize the above results by deriving closed-form expressions for both

$$\int_0^b \left[\sinh^{-1}(\cosh u) - u\right] du, \quad b \geq 0, \quad \text{and} \quad \int_{\alpha/2}^{\beta/2} \ln(\tanh z) dz, \quad b \in \mathbb{R},$$

where $\beta := b + \sinh^{-1}(\cosh b)$. I then use these results to derive a functional relation for the dilogarithm function valid for all $b \geq \alpha/2$.

†Note that $\mathrm{Li}_2(1) = \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$, in agreement with Euler's result.