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Quasi-Periodic Solutions of the Generalized KdV Equation

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Abstract. This paper concerns the existence of real analytic quasi-periodic solutions close to the constant function 1 of the generalized KdV equation. The proof is based on an abstract KAM theorem for infinite dimensional Hamiltonian systems.

Key Words: KAM theorem, generalized KdV equation, normal form, quasi-periodic solution.

AMS Subject Classifications: 37K55, 35B15

1 Introduction

Consider the following equation on $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$:

$$u_t=-u_{xxx}+(2p+2)(u^{2p+1})_x. \tag{1.1}$$

When $p\in\mathbb{Z}^+\backslash\{1\}$, the equation is called the gKdV equation (generalized KdV equation).

Eq. (1.1) is a Hamiltonian system with Hamiltonian

$$H(u)=\frac{1}{2}\langle Au,u\rangle+\int_0^{2\pi}u^{2p+2}dx,\quad A=-\partial_{xx}. \tag{1.2}$$

Denote d as the order of linearity's space derivative and δ as the order of nonlinearity's space derivative.

When $\delta\le 0$, the nonlinearity is called bounded perturbation. There are so many results which concern the existence of quasi-periodic solution of PDEs with bounded perturbation, see [5–8,11–13,16,20–22,25].

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When $0 < \delta \leq d-1$, the KAM theorem's keypoint is Kuksin's lemma, see [17]. Kuksin's lemma is used to prove the KdV equation's persistence of time quasi-periodic solutions. See also Kappeler and Pöschel in [15]. Liu and Yuan prove a generalized Kuksin's lemma and construct KAM theorem for a class of derivative nonlinear Schrödinger equations, see [18,19].

When $\delta \geq d$, Baldi, Berti and Montalto prove KAM theorems for the perturbed KdV equation and mKdV equation, see [1–3]. Baldi, Berti, Haus and Montalto [4] prove the KAM theorem for the gravity water waves equation in finite depth. Chen and Geng [9,10] establish linearly stable KAM tori for higher dimensional Kirchhoff equations and one dimensional forced Kirchhoff equations.

In [23], Sun use a KAM theorem for the generalized KdV equation to prove the existence of quasi-periodic solutions of the generalized KdV equation. But the resonance cases are too complicated due to the large number p in (1.1). Thus Sun only prove the existence of periodic solutions of the generalized KdV equation. Recently, we notice that one can assume the solutions expand at a constant not 0. Without loss of generality, let u close to the constant 1 satisfy

$$u(t,x)=\sum_{n\neq 0}\gamma_n q_n(t)\phi_n(x)+1,$$

the corresponding eigenfunctions

$$\phi_n(x)=(2\pi)^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}nx},\quad \gamma_n=\sqrt{2|n|}.$$

Take u into

$$\int_0^{2\pi} u^{2p+2}dx,$$

the constant 1 will bring the second order terms

$$\sum_{n\geq 1}(2p+2)(2p+1)n|q_n|^2,$$

the third order terms

$$\frac{(2p+2)(2p+1)(2p)}{6\sqrt{2\pi}}\sum_{i+j+k=0}\gamma_i q_i\gamma_j q_j\gamma_k q_k,$$

the fourth order terms

$$\frac{(2p+2)(2p+1)(2p)(2p-1)}{48\pi}\sum_{i+j+k+l=0}\gamma_i q_i\gamma_j q_j\gamma_k q_k\gamma_l q_l$$

and so on. We only need to eliminate the third and fourth order terms to a normal form like $|q|_j^2|q|_k^2$ and treat other high order terms as perturbations. We have the following main result: