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Finding Similarity of Orbits Between Two Discrete Dynamical Systems via Optimal Principle

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Abstract. Whether there is a similarity between two physical processes in the movement of objects and the complexity of behavior is an essential problem in science. How to seek similarity through the adoption of quantitative and qualitative research techniques still remains an urgent challenge we face. To this end, the concepts of similarity transformation matrix and similarity degree are innovatively introduced to describe similarity of orbits between two complicated discrete dynamical systems that seem to be irrelevant. Furthermore, we present a general optimal principle, to give a strict characterization from the perspective of dynamical systems combined with optimization theory. For well-known examples of chaotic dynamical systems, such as Lorenz attractor, Chua's circuit, Rössler attractor, Chen attractor, Lü attractor and hybrid system, with using of the homotopy idea, some numerical simulation results reveal that a similarity can be found in rich characteristics and complex behaviors of chaotic dynamics via the optimal principle we presented.

AMS subject classifications: 37N40, 49K15, 65K10

Key words: Similarity, optimal principle, homotopy, discrete dynamical system, chaotic attractor.

1 Introduction

Discrete dynamical systems described by iteration of mappings appear everywhere, showing directive laws from physical science or result from simulations to better understand differential equations numerically. Generally, it is much more difficult but interesting to investigate how complex behavior happens to discrete dynamical systems than continuous dynamical systems after some iterations, since there are probably greater covered

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ranges and more ghost phenomena. Along with the development of computer technology, modeling problems by means of discrete dynamical systems mathematically has already been gained in different fields such as biology, economics, demography, engineering, and so on. It is universally acknowledged that no matter how different the various technologies develop as well as the objects appear in the research process, there are certain underlying similarities.

Similarity, in addition to being frequently encountered, is viewed as a fundamental concept in scientific research. The idea of similarity has gained widespread popularity in the era of big data and machine learning by various means. For instance, scale similarity is found in many natural phenomena in the universe [1]. An embedding-based vehicle method with deep representation learning drastically accelerates trajectory similarity computation [2]. A novel brain electroencephalography (EEG) clustering algorithm not only handles the problem of unlabeled EEG, but also avoids the time-consuming task of manually marking the EEG [3]. Based on the cosine similarity, a transductive long short-term memory model is developed for temperature forecasting [4]. Self-similar coordinates are investigated in Lattice Boltzmann equation, showing that the time averaged statistics for velocity and vorticity express self-similarity at low Reynolds [5]. Many other applications include gene expression [6], image registration [7], web pages and scientific literature [8], fuzzy linguistic term sets [9], collaborative filtering [10], pattern analysis [11] and preferential attachment [12]. Indeed, the ubiquitous similarity is attributed to facilitate prediction of indeterminate events by analyzing known data, being an essential task in many natural systems and phenomena of real life.

A core part of similarity search is the so-called similarity measure whose famous characteristic is able to assess how similar two sequences are, in other words, the degree to which a given sequence resembles another. Many researchers have paid great attention to devise a proper similarity measure and have achieved several valuable results, which can be roughly categorized into two sorts. One sort is based on the traditional measures, such as Euclidean distance, dynamic time warping, cosine and cotangent similarity measures and Pearson correlation coefficient [13]. The other sort is some transform-based methods, such as singular value decomposition, principal component analysis, Fourier coefficients, auto-regression and moving average model [14, 15]. The cautious selection of similarity measure scheme has long been a research hotspot, affecting the accuracy of further data mining tasks directly, such as classification, clustering and indexing [16, 17].

Up to now, whether there is a similarity between two physical processes and how to seek the similarity through a mathematical principle are still remain a significant challenge. Several theoretical approaches are available to deal with this problem by taking into account asymptotic equivalence, synchronization and stability just as some kind of similarity. Furthermore, almost all similarity measure criteria are exploited according to application background and actual data, which can only be regarded as quantitative representations to estimate pairwise similarity of a given series resembles another under certain conditions.

Determining a similarity between orbits derived from chaotic systems generally char-