Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2024-0132

Vol. **37**, No. 3, pp. 675-700
March 2025

A Four-State HLL Riemann Solver for Numerical Simulation of Magneto-Hydrodynamics Based on the Least Squares Solution for the Middle Wave

Xinyue Xi[1,2], Xiaocheng Guo[1,2,*] and Chi Wang[1,2]

[1] *Key Laboratory of Solar Activity and Space Weather, National Space Science Center, Chinese Academy of Science, Beijing, 100190, China.*
[2] *College of Earth and Planetary Sciences, University of Chinese Academy of Sciences, Chinese Academy of Science, Beijing, 100049, China.*

Communicated by Kun Xu

Received 8 June 2024; Accepted (in revised version) 26 October 2024

Abstract. By applying the least squares solution for the middle wave analysis inside the Riemann fan, we construct a four-state Harten-Lax-van Leer (HLL) Riemann solver for numerical simulation of magneto-hydrodynamics (MHD). First, we revisit the two-state HLL scheme and obtain the two outer intermediate states across the two out-bounding fast waves through Rankine-Hugoniot (R-H) conditions; Second, the two inner intermediate states are calculated using a geometric interpretation of the R-H conditions across the middle contact wave. This newly constructed four-state HLL solver contains different wave structures from those of the HLLD Riemann solver; namely, the two Alfvén waves are replaced as the two combination waves originated from the merging of Alfvén and slow waves inside the Riemann fan. As we tested, this solver resolves the MHD discontinuities well, and has better capture ability than the HLLD solver for the slow waves, although it appears more diffusive than the latter in the situations where the slow waves are not solely generated. Overall, the new solver has the similar accuracy as the HLLD solver, thus it is suitable for the calculation of numerical fluxes for the Godunov-type numerical simulation of MHD equations where the slow waves are expected to be resolved.

AMS subject classifications: 65M08, 85-08, 76W05, 86A04

Key words: Magnetohydrodynamics, Riemann solver, HLLD, HLLC-Linde.

---

*Corresponding author. *Email addresses:* `xyxi@swl.ac.cn` (X. Xi), `xcguo@swl.ac.cn` (X. Guo), `cw@spaceweather.ac.cn` (C. Wang)

1 Introduction

Magnetohydrodynamics (MHD) equations are extensively applied in various fields, such as laboratory, space, and astrophysical plasmas, and the related research has made remarkable progress in the last century. They can be characterized as partial differential equations with hyperbolic conservation laws, however, it is difficult to obtain analytical solutions directly. In practice, numerical methods are commonly used for solution.

Finite-volume methods (FVM) have been developed as a leading computational approach for solving systems of hyperbolic conservation laws. Specifically, Godunov's method [18] and its derivatives have gained increasing popularity for the Euler equations of hydrodynamics (HD) due to their robustness and their ability to achieve high resolution near discontinuities. In the last few decades, the application of Godunov methods has been extended from HD to MHD systems. As variables are defined at the cell center and fluxes are constructed at the cell interface, solving the Riemann problem becomes essential to update the fluxes at each time step in time-dependent simulations. However, solving the Riemann problem exactly is computationally expensive. Therefore, approximate Riemann solvers have been developed and extensively utilized over the past three decades [3, 10, 14, 30].

The Roe-type approximate Riemann solver initially applied in HD applications [32] was later extended to MHD simulations [2, 10, 11]. This solver includes all MHD wave modes and has proven to be accurate in numerous MHD applications. However, it has been noted that this linearized solver does not maintain the positivity for density or pressure. Moreover, it involves a complex and time-consuming eigen-decomposition process. This challenge is further exacerbated in low-β MHD cases, then switching to the Harten-Lax-van Leer(HLL)-type approximate Riemann solver is necessary to maintain positivity.

The HLL-type Riemann solver was initially proposed by Harten et al. [22], and is widely used for its effective preservation of positivity [40]. As one typical example, HLLE solver (where 'E' stands for Einfeldt) follows entropy and positivity conditions by using suitable wave-speed bounds for the conservative hyperbolic system [16]. Unlike the Roe-type solver, it is more straightforward to implement and does not require eigen-decomposition. However, its limitation lies in assuming only one intermediate state between two acoustic (or magneto-acoustic) waves, resulting in excessive diffusion and poor resolution of isolated contact discontinuities. To address this, the HLLEM solver (where 'M' stands for Macherner) was introduced to incorporate anti-diffusion terms through partial eigen-decomposition [16]. This approach, extended to MHD, adds anti-diffusion terms to all waves except the two fast magnetosonic waves [38]. Another strategy involves introducing a discontinuity in the intermediate state, resulting in two intermediate states [22].

The HLLC (with 'C' representing Contact discontinuity) Riemann solver is an extension of the HLL-type Riemann solver. Originating from HD, Toro et al. [37] initially posited the existence of a contact discontinuity while using the HLL solver. They assumed that, under this premise, the two intermediate states share the same velocity and