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Mortar Finite Element Method for the Coupling of Time Dependent Navier-Stokes and Darcy Equations

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Abstract. The article discusses a nonlinear system that is dependent on time and coupled by incompressible fluid and porous media flow. Treating Darcy flow as dual-mixed form, we propose a variational formulation and prove the well-posedness of weak solutions. The discretization of domain is accomplished using a triangular mesh, with the lowest order Raviart-Thomas element utilized for Darcy equations and Bernardi-Raugel element used for Navier-Stokes equations. Using the mortar method, we construct the spaces from which numerical solutions are sought. Based on backward Euler method, we establish a fully discrete algorithm. At each single time level, the first-order convergence is demonstrated through the use of the Gronwall inequality. Numerical experiments are provided to illustrate the algorithm's effectiveness in approximating solutions.

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Key words: Darcy equations, Navier-Stokes equations, coupling, finite elements, mortar.

1 Introduction

Over the last two decades, the coupling model has received widespread attention due to its numerous applications in fluid mechanics. In brief, the model is formed by two parts. One is an incompressible fluid with viscosity and the other is a porous media saturated by same fluid. For instance, water in rivers or lakes is the fluid, while the riverbed or the bottom of lakes made up of sands and rocks constitutes the porous media. In free fluid region, motion of flow is governed by Navier-Stokes equations which is derived from the conservation of momentum, with the incompressible condition being that the fluid velocity is divergence-free. The porous body is assumed to be made up of rigid solids

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with no deformation; therefore, for the fluid inside the porous body, the balance of linear momentum can be described as Darcy equations. The coupling model can simulate and forecast the progression of surface water percolating through the riverbed or the bottom of lakes, then diffusing and transporting substances (such as contaminants, salt) with groundwater.

There exists an interface between free fluid and porous media flow. This interface couples Navier-Stokes equations and Darcy flow through three conditions. The first condition is the continuity of velocity in normal direction that arises from the incompressibility of fluid. The second condition is the balance of normal forces. The third condition pertains to the velocity in tangential direction. Beavers and Joseph [7] propose a law demonstrating that the difference between the slip of free fluid and the tangential velocity is proportional to the sheer rate of free fluid with a constant equal to the square root of permeability. Furthermore, [8] shows that the tangential velocity of Darcy equations on the interface is much smaller than other quantities. Thus, Saffman drops it to modify the Beavers-Joseph law. In [9], Jäger and Mikelić present a rigorous mathematics proof for Saffman's form. It is justified that Beavers-Joseph-Saffman law is valid with the pore size of porous medium tending to zero. These interface conditions lead to a well-posed boundary value problem. After that, many scholars propose a large number of algorithms to solve the coupling model.

The beginning are [10] and [27] for the stationary coupling. Discacciati, Miglio and Quarteroni [10] treat Darcy equations as a Poisson problem which is another description for the motion of fluid in porous media. In the Poisson problem, one needs to determine piezometric head, which is the sum of pressure and the elevation from a reference level. Ignoring the elevation, only the pressure remains unknown. The velocity is then obtained by multiplying the gradient of pressure with the rock permeability tensor. Given that the coupling model describes multiphysics phenomena and consists of different partial differential equations, a logical and obvious choice would be to use the domain decomposition method [11–15] to solve Stokes and Poisson equations separately. Another way decoupling the model is two grid method [16,17] which computes the coupling system only on coarse grid. It is well-known that the difference between Stokes and Navier-Stokes equations lies in the nonlinear convection term, which poses significant challenges in analysis and implementation of schemes. To tackle this nonlinear system, the above-mentioned decoupling methods [18, 19] and discontinuous Galerkin method [20] have been studied for the coupling of Navier-Stokes and Darcy equations. Our focus is on the non-stationary problem which is of fundamental importance. For the time dependent coupling, all of algorithms [21–26] are based on the primal form of Darcy equations, which is the same as [10].

Compared with [10], Layton, Schieweck and Yotov [27] adopt dual-mixed weak formulation for Darcy equations in which the velocity with no accuracy losing and the pressure are computed at the same time. For the balance of normal forces on the interface, Langrange multiplier is introduced by which flux continuity is imposed on the space for