Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2023-0213

Vol. **37**, No. 3, pp. 783-809
March 2025

An Adaptive Polygonal Finite Volume Element Method Based on the Mean Value Coordinates for Anisotropic Diffusion Problems

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Received 10 August 2023; Accepted (in revised version) 21 November 2023

Abstract. We propose a polygonal finite volume element method based on the mean value coordinates for anisotropic diffusion problems on star-shaped polygonal meshes. Because the convex cells with hanging nodes are always star-shaped, the computation on them is no longer a problem. Naturally, we apply this advantage of the new polygonal finite volume element method to construct an adaptive polygonal finite volume element algorithm. Moreover, we introduce two refinement strategies, called quadtree-based refinement strategy and polytree-based refinement strategy respectively, and they all have great performance in our numerical tests. The new adaptive algorithm allows the use of hanging nodes, and the number of hanging nodes on each edge is unrestricted in general. Finally, several numerical examples are provided to show the convergence and efficiency of the proposed method on various polygonal meshes. The numerical results also show that the new adaptive algorithm not only reduces the computational cost and the implementation complexity in mesh refinement, but also ensures the accuracy and convergence.

AMS subject classifications: 65N08, 65N12

Key words: Polygonal finite volume element method, mean value coordinates, hanging nodes, adaptive algorithm.

1 Introduction

With the development of science and engineering computations, the research of finite volume element method (FVEM) has made great progress in the past several decades,

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see, e.g. [1–3]. FVEM is usually regarded as a special type of finite volume methods, where the solution space is the same as that of the finite element method, and it also has the local conservation property like other finite volume methods. For the 2D problems, lots of finite volume element schemes have been proposed, such as linear schemes on triangular meshes (e.g. [4–8]), linear schemes on quadrilateral meshes (e.g. [9–12]), higher-order schemes on triangular meshes (e.g. [13–18]), higher-order schemes on quadrilateral meshes (e.g. [19–24]) and so on, for incomplete references. Recently, a polygonal finite volume element method based on the Wachspress coordinates (PFVEM$-$WP) was suggested in [25] for anisotropic diffusion problems on convex polygonal meshes, and an optimal H^1 error estimate was obtained under the coercivity assumption. After that, the authors of [26] proposed a polygonal finite volume element method by imposing a certain postprocessing technique on a polygonal finite element method, and proved the existence, uniqueness, optimal H^1 and L^2 error estimates of the post-processed solution. For the higher-order schemes on convex polygonal meshes, a family of quadratic serendipity polygonal finite volume element method for anisotropic diffusion problems was constructed and analyzed in [27]. We remark that the above polygonal finite volume element methods can only be implemented on strictly convex polygonal meshes. In this article, we propose a new polygonal finite volume element method for solving anisotropic diffusion problems on star-shaped polygonal meshes with possibly non-convex cells or hanging nodes.

We point out that triangular or quadrilateral cells with hanging nodes can be viewed as a special type of star-shaped polygonal ones. Keeping this fact in mind, we can use the new finite volume element method to construct a novel adaptive algorithm. To this end, let us briefly review some relevant developments of adaptive algorithms. In the scientific computation and engineering simulation, there are lots of partial differential equation problems that need to capture steep gradients, discontinuities, singularities and so on. At this moment, uniform meshes are not the best and economical choice. Adaptive strategy is a popular and effective method to solve these problems, e.g. [28–49], which can automatically adjust local mesh refinement, mesh redistribution or order enrichment to obtain the expected accuracy. In practice, the usual and popular adaptive methods include h-adaptive method (mesh refinement), r-adaptive method (mesh redistribution), p-adaptive method (order enrichment) and hp-adaptive method (the combination of h-adaptive method and p-adaptive method). In this paper, we focus on the h-adaptive method, which has many mesh refinement strategies, such as newest vertex bisection [28, 29], longest edge bisection [30, 31], quadtree-based refinement strategy [32–34] and polytree-based refinement strategy [35, 36] and so on. These refinement strategies usually generate hanging nodes after each refinement, due to the mesh level mismatch between adjacent cells. In recent decades, there are many methods and techniques to solve this problem, such as the further refinement for the cells with hanging nodes (or called "completion procedure") [37–39], the construction of the connectivity mappings to constrain hanging nodes [40], the Lagrange multiplier method [41, 42] and so on. The hanging-node problem also exists in the adaptive finite volume methods, see