Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2024-0112

Vol. **37**, No. 2, pp. 289-314
February 2025

Learning PDEs from Data on Closed Surfaces with Sparse Optimization

Zhengjie Sun[1], Leevan Ling[2] and Ran Zhang[3,*]

[1] *School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing, China.*
[2] *Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Hong Kong.*
[3] *School of Mathematics, Shanghai University of Finance and Economics, Shanghai, China.*

Received 13 May 2024; Accepted (in revised version) 14 September 2024

Abstract. Discovering underlying partial differential equations (PDEs) from observational data has important implications across fields. It bridges the gap between theory and observation, enhancing our understanding of complex systems in applications. In this paper, we propose a novel approach, termed physics-informed sparse optimization (PIS), for learning surface PDEs. Our approach incorporates both L_2 physics-informed model loss and L_1 regularization penalty terms in the loss function, enabling the identification of specific physical terms within the surface PDEs. The unknown function and the differential operators on surfaces are approximated by some extrinsic meshless methods. We provide practical demonstrations of the algorithms including linear and nonlinear systems. The numerical experiments on spheres and various other surfaces demonstrate the effectiveness of the proposed approach in simultaneously achieving precise solution prediction and identification of unknown PDEs.

AMS subject classifications: 41A05, 65D12

Key words: Meshless methods, data-driven modeling, sparse optimization, surface PDE.

---

1 Introduction

Data-driven modeling only with available data has been widely considered in learning theory and its associated application areas. While many phenomena in science and engineering can be formulated as partial differential equations (PDEs), traditional PDE models predominantly rely on system behavior descriptions and classical physical laws.

*Corresponding author. *Email addresses:* zhengjiesun@njust.edu.cn (Z. Sun), lling@hkbu.edu.hk (L. Ling), zhang.ran@mail.shufe.edu.cn (R. Zhang)

Consequently, the fundamental challenge lies in distilling the underlying PDEs from the given data.

The methodology of data-driven modeling can even date back to the time of Kepler, who employed the most meticulously guarded astronomical data of the day to discover a data-driven model for planetary motion. Later in 1975, Gauss introduced the least squares regression (LSR) algorithm, which provided a numerical framework for learning underlying models from data. The classical Prony's method was originally developed to use the difference equation as a discrete analogue of the linear ordinary differential equation, then used LSR to compute coefficients of linear ordinary difference equations. However, it is known to perform poorly in the presence of noisy samples. To address this limitation, various stabilization and modification methods for Prony's method have been proposed, as discussed in Osborne et al. [22] and Zhang et al. [40], for example. Furthermore, Schmidt and Lipson in [31] introduced symbolic regression and evolutionary algorithms to directly learn physical laws, such as nonlinear energy conservation laws and Newtonian force laws, from experimentally captured data.

With the rapid advancement of data storage and data science tools, data-driven discovery of potential models has entered a new era with the emergence of big data. Numerical simulations including Lorentz system (ODE) and fluid flow (PDE) are presented in [7] with thresholded least square method to promote sparsity. A deep PDE network has been considered to deal with the multi-dimensional systems based on the convolution kernels in [21]. [27, 32] proposed the physics-informed neural networks (PINN) framework to address both data-driven solutions and data-driven discovery of PDEs. Wu et al. [38] successfully applied the sparse regression to learn a chaotic system, effectively simulating the sustaining oscillations during the sedimentation of a sphere through non-Newtonian fluid. Wu et al. [39] further discussed two probabilistic solutions for the dynamical system, namely the random branch selection iteration (RBSI) and random switching iteration (RSI). Other related works include, for instance, parameter identifications [15] in uncertainty quantification and calibration [18] in the financial industry, which often involve more prior knowledge about the corresponding PDE models.

In recent years, surface PDEs find a wide range of applications in various fields, including imaging processes, biological reactions, fluid dynamics, and computer graphics [1, 4, 6, 10]. As a result, there has been significant research on numerical methods to solve surface PDEs, employing various techniques such as intrinsic, extrinsic, and embedding methods. Intrinsic methods involve local parameterization of surfaces and discretization of surface differential operators on surface meshes [12]. Extrinsic methods transform the differential operators on surfaces into extrinsic coordinates [8], while embedding approaches extend the surface PDEs to embedding spaces [24]. Although there have been advancements, such as the use of physics-informed convolutional neural networks (PICNN) to solve PDEs on spheres [20], the inverse problem of discovering hidden PDEs based on surface data remains an open challenge. In this paper, our focus is on addressing this challenge by developing novel techniques for the discovery of nonlinear PDEs on closed surfaces.