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Convergence Analysis of PINNs with Over-Parameterization

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Abstract. Recently, physics-informed neural networks (PINNs) have been shown to be a simple and efficient method for solving PDEs empirically. However, the numerical analysis of PINNs is still incomplete, especially why over-parameterized PINNs work remains unknown. This paper presents the first convergence analysis of the over-parameterized PINNs for the Laplace equations with Dirichlet boundary conditions. We demonstrate that the convergence rate can be controlled by the weight norm, regardless of the number of parameters in the network.

AMS subject classifications: 65M15, 65N15, 65Y20

Key words: PINNs, over-parameterization, convergence rate, deep approximation with norm control.

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1 Introduction

Classical numerical methods, such as finite element methods [11,13], face significant challenges when applied to high-dimensional PDEs. In contrast, the success of deep learning techniques in high-dimensional data analysis has paved the way for promising methods based on deep neural networks to tackle high-dimensional PDEs, gaining substantial attention in recent years [2,9,19,32,43,45,50,52]. Due to their remarkable approximation power, deep neural networks have led to the development of various numerical

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schemes, such as the deep Ritz method (DRM) [50] and PINNs [43]. While DRM is primarily designed for PDEs in variational form, PINNs are based on residual minimization [2, 32, 43, 45], making them highly adaptable to a wide variety of PDEs [24, 41, 42]. This flexibility has further expanded the application of PINNs to complex and nonlinear PDEs, marking a significant advancement in numerical methods for high-dimensional problems.

Recently, researchers have undertaken extensive mathematical investigations to understand the underlying principles of DRM and PINNs [16,21,22,25,28,29,33–39,44,46,48, 49, 51]. Notably, these analyses have predominantly focused on scenarios where training sample quantities exceed neural network parameters. However, practical implementations often favor over-parameterized networks due to their computational advantages. It has been demonstrated that (stochastic) gradient descent with randomized initialization and small step sizes can achieve linear convergence in over-parameterized settings, even though the underlying optimization problem is highly non-convex [1, 12, 15, 23, 31, 53]. The efficacy of over-parameterized networks remains theoretically enigmatic, presenting a fundamental challenge in deep learning research [4,5,10]. While the double descent phenomenon has provided insights into over-parameterization in linear and kernel models [3–8, 30, 40, 47], contradictory findings exist, such as [27]'s demonstration of inconsistent empirical risk minimization in over-parameterized networks for nonparametric regression.

In this work, we analyze the convergence rate of PINNs with over-parameterization for Laplace equations with Dirichlet boundary conditions. The analysis begins by decomposing the total error into approximation and statistical errors, following the methodology in [16,26,34]. The approximation error is addressed through results in Sobolev spaces with norm constraints. While a similar approach was used in [14], it is important to note that [14] focuses on the deep Ritz method for solving second-order elliptic equations with Robin boundary conditions, whereas our study analyzes PINNs for Laplace equations with Dirichlet boundary conditions. To ensure completeness, we have restated this approximation error control method within the current paper. Additionally, the statistical error is controlled via the weight norms by bounding the Rademacher complexity of the neural network class. By appropriately tuning the weight constraints, we establish the final results that balance the approximation (bias) and statistical (variance) errors, provided that the number of parameters exceeds a certain threshold. The main contributions of this paper are as follows:

- We present a convergence analysis for over-parameterized PINNs in solving Laplace equations with Dirichlet boundary conditions, and provide guidelines for setting the network depth and width to achieve the desired accuracy.
- We establish an approximation error bound in H^2 Sobolev spaces with weight constraints.

The paper is structured as follows. Section 2 introduces neural network functions with norm constraints. In Section 3, we define the problem setup and present our main