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A Provably Positive-Preserving HLLD Riemann Solver for Ideal Magnetohydrodynamics. Part I: The One-Dimensional Case

Xiaoteng Zhang[1,2,4], Xun Wang[1,2], Zhijun Shen[3,4] and Chao Yang[1,2,*]

[1] *School of Mathematical Sciences, Peking University, Beijing 100871, P.R. China.*
[2] *PKU-Changsha Institute for Computing and Digital Economy, Hunan 410205, P.R. China.*
[3] *Institute of Applied Physics and Computational Mathematics, LCP, P.O. Box 8009-26, Beijing 100088, P.R. China.*
[4] *HEDPS, Center for Applied Physics and Technology, and College of Engineering, Peking University, Beijing 100871, P.R. China.*

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Abstract. Combining robustness and high accuracy is one of the primary challenges in the magnetohydrodynamics (MHD) field of numerical methods. This paper investigates two critical physical constraints: wave order and positivity-preserving (PP) properties of the high-resolution HLLD Riemann solver, which ensures the positivity of density, pressure, and internal energy. This method's distinctiveness lies in its ability to ensure that the wave characteristic speeds of the HLLD Riemann solver are strictly ordered. A provably PP HLLD Riemann solver based on the Lagrangian setting is established, which can be viewed as an extension of the PP Lagrangian method in hydrodynamics but with more and stronger constraint condition. In addition, the above two properties are ensured on moving grid method by employing the Lagrange-to-Euler transform. Meanwhile, a novel multi-moment constrained finite volume method is introduced to acquire third order accuracy, and practical limiters are applied to avoid numerical oscillations. Selected numerical benchmarks demonstrate the robustness and accuracy of our methods.

AMS subject classifications: 65M50, 76M12, 76W05

Key words: Approximate Riemann solver, positivity-preserving method, multi-moment constrained finite volume method, arbitrary-Lagrangian-Eulerian (ALE) framework.

*Corresponding author. *Email addresses:* `zxt2019@pku.edu.cn` (X. Zhang), `s151025@muc.edu.cn` (W. Wang), `shen_zhijun@iapcm.ac.cn` (Z. Shen), `chao_yang@pku.edu.cn` (C. Yang)

1 Introduction

The ideal magnetohydrodynamics (MHD) problems are frequently found in a large variety of applications, such as space weather, laboratory plasmas, and magnetically confined fusion plasma applications [20,22,32]. The numerical simulations of the aforementioned ideal MHD problems often require the ability to overcome harsh conditions, such as near vacuum and ultra-high density environments, where shocks of various forms and intensity often arise. Correspondingly, the robustness and accuracy of numerical algorithms are critical for solving ideal MHD problems.

Riemann solvers are regarded as critical components in the development of robust numerical methods for solving problems governed by hyperbolic conservation laws. With the advancement of computational magnetohydrodynamics, a series of Riemann solvers have been proposed under finite volume framework [1–3,7,8,12–14,17,21,23]. In particular, Brio and Wu [7], Dai and Woodward [12] have done some of the early development of Roe schemes for the MHD equations. The specific Roe linearization was extended to general cases by Cargo and Gallice [8], and Balsara [1]. But numerical simulations for low density plasmas can sometimes lead to unphysical solutions of negative densities or pressures. Several efforts have been made to reduce this risk.

Compared to linearized Riemann solvers, the HLL approximate Riemann solvers are more effective and reliable in general. Janhunen [16] has firstly proposed an MHD HLL Riemann solver with positivity-preserving (PP) property. Note that it might be necessary to make the wave speeds larger or smaller to guarantee positivity according to Janhunen's proof, but it is not clear how often this wavespeed adjustment should be applied. Wu et al. [38–41] directly give wave speeds that maintain the PP property in the MHD HLL Riemann solver, utilizing a novel equivalent formulation of the admissible state set along with very technical estimates, a technique also referred to as geometric quasilinearization [37].

A major issue of the HLL solver is that it is not able to effectively resolve contact waves. In order to enhance the resolution of contact discontinuity, Gurski [14] extended the HLLC Riemann solver of the Euler equations to the MHD. Since HLL/HLLC-type solvers may not exactly resolve rotational discontinuities due to the two-state approximation in the Riemann fan, Miyoshi and Kusano [21] developed a multi-wave HLLD scheme, which possesses high resolution nearly identical to the fully-wave Roe solver. They believed that confirming whether the PP property is satisfied or not in any situation is a difficult task. In order to simplify the discussion, only expansion waves were investigated. Bouchut et al. [5,6] constructed a multiwave approximate Riemann solver for 1D ideal MHD from a relaxation system and deduced necessary requirements for the solver to fulfill discrete entropy inequalities and the PP property. Minoshima et al. [23], Miyoshi and Kusano [24] presented an amended Riemann solver called LHLLD which is more robust than the origin HLLD solver [21,22]. There are also a series of highly nontrivial extensions [5,6,36] to move from a one-dimensional Riemann solver to high dimensions. Although the above-mentioned five-wave solvers have obtained very good