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A Class of Second-Order Energy-Stable Schemes for the Cahn-Hilliard Equation and Their Linear Iteration Algorithm

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Abstract. In this paper, we study a class of second-order accurate and energy-stable numerical schemes for the Cahn-Hilliard model. These schemes are constructed by combining the Crank-Nicolson approximation with three stabilization terms in time and employing the Fourier pseudo-spectral method in space. This class of schemes includes the second-order schemes presented in previous works while providing new schemes by introducing stabilization terms. To solve these schemes with strong nonlinearity efficiently, we propose a linear iteration algorithm and prove that the algorithm satisfies a contraction mapping property in the discrete l^4 norm. Furthermore, we establish a comprehensive theoretical analysis, including unique solvability, mass conservation, energy stability, and convergence based on a uniform-in-time l^∞ bound of the numerical solution for the proposed second-order scheme. Some numerical simulation results with many different sets of stabilization parameters are presented to conclude the paper.

AMS subject classifications: 35Q99, 65M06, 65M12, 74A50

Key words: Cahn-Hilliard equation, second-order energy-stable scheme, Fourier pseudo-spectral method, energy stability, linear iteration.

1 Introduction

In this paper, we consider the Cahn-Hilliard (CH) equation

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$$\partial_t \Phi = \Delta \mu \quad \text{with} \quad \mu := \frac{\delta E[\Phi]}{\delta \Phi}, \tag{1.1}$$

where $E[\Phi]$ is the free energy functional of Ginzburg-Landau type,

$$E[\Phi] = \int_\Omega \left(\frac{\epsilon^2}{2} |\nabla \Phi|^2 + F(\Phi) \right) \mathrm{d}\mathbf{x} \quad \text{with} \quad F(\Phi) := \frac{1}{4}(\Phi^2 - 1)^2. \tag{1.2}$$

Here, the real-valued function Φ represents the concentration difference in a binary system, where the spatial domain is denoted as $\mathbf{x} \in \Omega \subseteq \mathbb{R}^d (d = 2 \text{ or } d = 3)$ and $\epsilon > 0$ is an interface width parameter. The CH equation (1.1) is a model for continuously describing the phase separation process of binary mixtures [4] and has been widely used in many fields such as the physical and materials sciences [1,26], multi-phase fluid dynamics [20], biology simulations [30] and image inpainting processing [3].

Throughout this paper, we adopt periodic boundary conditions for both Φ and chemical potential μ, along with the initial condition $\Phi(\mathbf{x},0) = \Phi_0(\mathbf{x})$, $\mathbf{x} \in \Omega$. In particular, due to the gradient structure of (1.1), two most important properties of the CH model (1.1) are the mass conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi, 1) = 0,$$

and the energy dissipation law

$$\frac{\mathrm{d}}{\mathrm{d}t} E[\Phi] = \left(\frac{\delta E}{\delta \Phi}, \partial_t \Phi \right) = (\mu, \Delta \mu) = -\|\nabla \mu\|^2 \leq 0,$$

where $(\cdot,\cdot)$ and $\|\cdot\|$ denote the $L^2(\Omega)$ inner product and the associated norm, respectively.

Along the numerical front of structure-preserving algorithms, many researchers have focused on developing energy-stable schemes, including the convex splitting methods [15,16], the stabilizing term techniques [22,24,35], the fully implicit structure-preserving schemes [14,17], the exponential time difference method [21] and the energy quadratization approaches such as IEQ, SAV and their variants [28,32,33]. It is worth noting that the convex splitting method is relatively perfected in theoretical analysis due to its natural preservation of unique solvability and energy stability. The basic idea is to split the energy functional into a combination of convex and concave functionals. The time-discretized schemes are designed by treating the convex and concave parts implicitly and explicitly, respectively. Bailo et al. [2] combined the upwind methodology with the classical convex splitting approach in time and finite-volume schemes in space for the CH equation to construct a first-order scheme which unconditionally and discretely preserves the boundedness of the phase field and the dissipation of the free energy. For the three-dimensional CH equation, Guo et al. [19] constructed a second-order scheme,