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The Effective Use of BLAS Interface for Implementation of Finite-Element ADER-DG and Finite-Volume ADER-WENO Methods

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Abstract. Numerical methods of the ADER family, in particular finite-element ADER-DG and finite-volume ADER-WENO methods, are among the most accurate numerical methods for solving quasilinear hyperbolic PDE systems. The internal structure of ADER-DG and ADER-WENO numerical methods contains a large number of basic linear algebra operations related to matrix multiplications. The main interface of software libraries for matrix multiplications for high-performance computing is BLAS. An effective method for integration the standard functions of the BLAS interface into the implementation of these numerical methods is presented. The calculated matrices are small matrices; and this allows to use effectively JIT technologies. The proposed approach immediately operates on AoS, which allows to efficiently calculate flux, source and non-conservative terms without transposition. The obtained computational costs demonstrated that the effective implementation, based on the use of the JIT functions of the BLAS, outperformed both the implementation based on the general BLAS functions and the vanilla implementations by several orders of magnitude. The complexity of developing an implementation based on the proposed approach does not exceed the complexity of developing a vanilla implementation. Performance analysis using roofline partly explains the observed features of the decreasing of computational costs.

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1 Introduction

Numerical methods of the ADER family, including finite-element discontinuous Galerkin methods (ADER-DG) and finite-volume methods based on WENO reconstruction

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(ADER-WENO), are modern accurate methods of arbitrarily high order intended for solving systems of quasilinear partial differential equations (PDE systems). Titarev and Toro [1,2] developed the ADER paradigm for finite-volume methods, which was based on the solution of the generalized Riemann problem (GRP), and the evolution of the solution at finite-volume interfaces was determined by the Cauchy-Kovalevski procedure based on a high-order expansion of the solution in a Taylor series and the use of the original PDE system to calculate the derivatives (see also [3]). Further development of the ADER paradigm for finite-volume methods was carried out in the works of Titarev and Toro [4–6]. Dumbser *et al.* [7,8] modified the ADER paradigm for finite-volume methods based on the use of a discrete space-time solution obtained in a separate stage of the local space-time DG-predictor (LST-DG predictor), and showed higher solution accuracy than the original version, especially for stiff problems.

Finite-element ADER-DG methods are fundamentally linear methods for solving PDE systems, so they are subject to the well-known Godunov theorem and such methods require limiters to preserve the monotonicity of the numerical solution. Currently, finite-element ADER-DG and finite-volume ADER-WENO methods are based on the development of the Multi-dimensional Optimal Order Detection (MOOD) paradigm [9–12], which resulted in the development [13,14] of finite-element ADER-DG methods with *a posteriori* correction of the solution in subcells by a finite-volume limiter, which can be the finite-volume ADER-WENO method [15,16]. This approach made it possible to preserve the monotonicity of the numerical solution and the subgrid resolution characteristic of DG methods [13]. However, the further development of purely finite-volume methods of the ADER family, carried out in the works [17–23], is not directly related to the ADER-DG method with a posteriori correction of the solution.

Currently, the finite element ADER-DG methods and the finite volume ADER-WENO methods are used to solve a wide range of specific problems in the physics and mechanics of continuous media. The main application, in this case, is associated with the use of the finite-element ADER-DG methods with a posteriori correction of the solution in subcells by a finite-volume limiter, which can be the finite-volume ADER-WENO method. Among these problems, it is necessary to highlight the problems of simulating ideal and dissipative flows in both classical hydrodynamics and magnetohydrodynamics [24–26], as well as special and general relativistic hydrodynamics and magnetohydrodynamics [27–29], the latter of which are characterized by high stiffness of dissipative terms. It is also necessary to note the works in which these numerical methods were used to solve problems of the motion of elastic deformable media [31–33], problems of seismic wave propagation [31,34–36], solving shallow water equations [39–41], simulating blood flow [42], simulating compressible barotropic two-fluid flows [37] and nonlinear dispersive systems [38]. A special place is occupied by problems associated with the study of the formation and propagation of detonation waves in reacting flows, which are characterized by abnormally high stiffness, for which the finite element ADER-DG methods and finite volume ADER-WENO methods were developed and studied in the works [43,44]. Application of finite element ADER-DG methods and finite volume ADER-WENO meth-