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Robust Decoding from Binary Measurements with Cardinality Constraint Least Squares

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Abstract. The principal goal of 1-bit compressive sampling is to decode n-dimensional signals with a sparsity level of s from m binary measurements. This task presents significant challenges due to nonlinearity, noise, and sign flips. In this paper, we propose the use of the cardinality-constrained least squares decoder as an optimal solution. We establish that, with high probability, the proposed decoder achieves a minimax estimation error, up to a constant c, as long as $m \geq \mathcal{O}(s\log n)$. In terms of computational efficiency, we employ a generalized Newton algorithm (GNA) to solve the cardinality-constrained minimization problem. At each iteration, this approach incurs the cost of solving a least squares problem with a small size. Through rigorous analysis, we demonstrate that, with high probability, the ℓ_∞ norm of the estimation error between the output of GNA and the underlying target diminishes to $\mathcal{O}(\sqrt{\frac{\log n}{m}})$ after at most $\mathcal{O}(\log s)$ iterations. Furthermore, provided that the target signal is detectable, we can recover the underlying support with high probability within $\mathcal{O}(\log s)$ steps. To showcase the robustness of our proposed decoder and the efficiency of the GNA algorithm, we present extensive numerical simulations and comparisons with state-of-the-art methods.

AMS subject classifications: 65D18, 68U05, 68U07

Key words: 1-bit compressive sampling, least square with cardinality constraint, minimax estimation error, generalized Newton algorithm, support recovery.

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1 Introduction

Compressive sensing is a formidable signal acquisition approach that allows for the recovery of signals beyond their bandlimitedness from noisy under-determined measurements. This method is particularly advantageous when the number of measurements is closer to the order of the signal complexity rather than following the Nyquist rate [9, 11, 12, 14]. In order to facilitate storage and transmission, quantization is necessary to convert infinite-precision measurements into discrete ones [37]. Among various quantization techniques, scalar quantization is commonly employed due to its low computational complexity. A scalar quantizer, denoted as $\mathcal{Q}(\cdot)$ with a bit depth of b, can be fully characterized by the quantization regions $\{[r_\ell, r_{\ell+1})\}_{\ell=1}^{L}$, which form a partition of $\mathbb{R}$. Here, $L=2^b$, $r_1=-\infty$, $r_{L+1}=\infty$, and the codebook $\{\omega_\ell\}_{\ell=1}^{L}$, where $\mathcal{Q}(t)=\omega_\ell$ if $t\in[r_\ell, r_{\ell+1})$. One of the extreme cases of scalar quantization is the 1-bit quantizer, denoted as $\mathcal{Q}(t)=\mathrm{sign}(t)$. This quantizer codes the measurements into binary values using a single bit. It has been introduced into compressed sensing, giving rise to 1-bit compressed sensing (1-bit CS) [7]. The 1-bit CS has garnered significant attention due to its low hardware implementation and storage costs, as well as its robustness in scenarios with low signal-to-noise ratios [26].

1.1 Notation and 1-bit CS model

We denote by $\Psi_i\in\mathbb{R}^{m\times 1}$, $i=1,\cdots,n$, and $\psi_j\in\mathbb{R}^{n\times 1}$, $j=1,\cdots,m$, the ith column and jth row of Ψ, respectively. We denote zero vector by $\mathbf{0}$. We use $[n]$ to denote the set $\{1,\cdots,n\}$, and $\mathbf{I}_n$ to denote the identity matrix of size $n\times n$. For $A,B\subseteq[n]$ with cardinality $|A|,|B|$, $x_A=(x_i,\ i\in A)\in\mathbb{R}^{|A|}$, $\Psi_A=(\Psi_i,\ i\in A)\in\mathbb{R}^{m\times|A|}$ and $\Psi_{AB}\in\mathbb{R}^{|A|\times|B|}$ denotes a submatrix of Ψ whose rows and columns are listed in A and B, respectively. Let $x|_A=(x_i\mathbf{1}_{i\in A})\in\mathbb{R}^n$, where, $\mathbf{1}_A$ denotes the indicator function of set A. Let $|x|_{s,\infty}$ and $|x|_{\min}$ be the sth largest elements (in absolute value) and the minimum absolute value of x, respectively. We use $\mathcal{N}(\mathbf{0},\Sigma)$ to denote the multivariate normal distribution, with Σ symmetric and positive definite. Let $\gamma_{\max}(\Sigma)$ and $\gamma_{\min}(\Sigma)$ be the largest and the smallest eigenvalues of Σ, respectively. Let $\mathrm{supp}(x)$ denote the support of x. We use $\|x\|_\Sigma$ to denote the elliptic norm of x with respect to Σ, i.e., $\|x\|_\Sigma=(x^t\Sigma x)^{\frac{1}{2}}$. Let $\|x\|_p=(\sum_{i=1}^n|x_i|^p)^{1/p}$, $p\in[1,\infty]$, be the ℓ_p-norm of x. We denote the number of nonzero elements of x by $\|x\|_0$. The symbols $\|\Psi\|$ and $\|\Psi\|_\infty$ stands for the operator norm of Ψ induced by ℓ_2 norm and the maximum pointwise absolute value of Ψ, respectively. $\mathrm{sign}(\cdot)$ operates componentwise with $\mathrm{sign}(z)=1$ if $z\geq 0$ and $\mathrm{sign}(z)=-1$ otherwise, and $\odot$ denotes the pointwise Hadamard product. By $\mathcal{O}(\cdot)$, we ignore some positive numerical constants. 1-bit compressive sensing is even more challenging due to decode from noisy, nonlinear and sign-flipped measurements.

Following the works of Plan and Vershynin [32] and Huang and Jiao [20], we consider the 1-bit compressed sensing (CS) model described by the following equation:

$$y=\eta\odot\mathrm{sign}(\Psi x^*+\epsilon), \tag{1.1}$$