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Bound-Preserving Point-Average-Moment PolynomiAl-Interpreted (PAMPA) Scheme: One-Dimensional Case

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Abstract. We propose a bound-preserving (BP) Point-Average-Moment PolynomiAl-interpreted (PAMPA) scheme by blending third-order and first-order constructions. The originality of the present construction is that it does not need any explicit reconstruction within each element, and therefore the construction is very flexible. The scheme employs a classical blending approach between a first-order BP scheme and a high-order scheme that does not inherently preserve bounds. The proposed BP PAMPA scheme demonstrates effectiveness across a range of problems, from scalar cases to systems such as the Euler equations of gas dynamics. We derive optimal blending parameters for both scalar and system cases, with the latter based on the recent geometric quasi-linearization (GQL) framework of [Wu & Shu, *SIAM Review*, 65 (2023), pp. 1031–1073]. This yields explicit, optimal blending coefficients that ensure positivity and control spurious oscillations in both point values and cell averages. This framework incorporates a convex blending of fluxes and residuals from both high-order and first-order updates, facilitating a rigorous BP property analysis. Sufficient conditions for the BP property are established, ensuring robustness while preserving high-order accuracy. Numerical tests confirm the effectiveness of the BP PAMPA scheme on several challenging problems.

AMS subject classifications: 65M08, 76M12, 35L65, 35Q31, 65M12.

Key words: Point-Average-Moment PolynomiAl-interpreted (PAMPA) scheme, bound-preserving, convex limiting, Euler equations of gas dynamics, geometric quasilinearization (GQL).

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1 Introduction

We consider the one-dimensional hyperbolic partial differential equations (PDEs) of conservation laws:

$$\frac{\partial \mathbf{u}}{\partial t}+\frac{\partial \mathbf{f}(\mathbf{u})}{\partial x}=\mathbf{0} \tag{1.1}$$

with the initial condition $\mathbf{u}(x,0)=\mathbf{u}_0(x)$. Here, $x\in\Omega\subset\mathbb{R}$ is the spatial variable, $t\geq 0$ represents time, $\mathbf{u}(x,t)\in\mathbb{R}^n$ is an unknown vector function, and $\mathbf{f}(\mathbf{u})$ denotes the physical fluxes.

It is known that the entropy weak solution[†] of (1.1) often satisfies certain physical bounds. For instance, when (1.1) reduces to a scalar form, $u_t+f(u)_x=0$, the entropy solution of scalar conservation laws adheres to the maximum principle. Specifically, if $u_0(x)\in[\mathring{u}_{\min},\mathring{u}_{\max}]$, the entropy solution $u(x,t)$ will maintain these bounds for any $t>0$, remaining within the invariant domain $\mathcal{D}=[\mathring{u}_{\min},\mathring{u}_{\max}]$. In the case of a system of conservation laws, where $\mathbf{u}(x,t)\in\mathbb{R}^p$, one can often identify a convex invariant domain $\mathcal{D}\subset\mathbb{R}^p$ such that both $\mathbf{u}_0$ and $\mathbf{u}(x,t)$ lie within $\mathcal{D}$. In this work, we will focus on the one-dimensional compressible Euler equations, with

$$\mathbf{u}=\begin{pmatrix}\rho\\ m\\ E\end{pmatrix},$$

where ρ represents the fluid density, $m=\rho v$ is the momentum, v denotes the fluid velocity, and $E=e+\frac{1}{2}\rho v^2$ is the total energy, with e as the internal energy. The flux vector is given by

$$\mathbf{f}(\mathbf{u})=\begin{pmatrix} m\\ \frac{m^2}{\rho}+p\\ \frac{m(E+p)}{\rho}\end{pmatrix},$$

where $p=p(e,\rho)$ is the pressure, related to internal energy and density by an equation of state. For simplicity, we consider the ideal gas equation of state,

$$p=(\gamma-1)e,$$

where γ is a constant representing the specific heat ratio. For the Euler equations, a convex invariant domain $\mathcal{D}$ is defined by

$$\mathcal{D}=\left\{(\rho,m,E)\,\middle|\,\rho>0 \text{ and } e=E-\frac{m^2}{2\rho}>0\right\}. \tag{1.2}$$

It is highly desirable, and often crucial, to ensure that the numerical solution remains within the convex invariant domain $\mathcal{D}$. Numerical methods capable of preserving this

[†] An entropy weak solution should satisfy the following inequality in the distributional sense $\eta(\mathbf{u})_t+q(\mathbf{u})_x\leq 0$, where (η,q) is the so-called entropy pair.