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Inexact Block LU Preconditioners for Incompressible Fluids with Flow Rate Conditions

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Abstract. When studying the dynamics of incompressible fluids in bounded domains the only available data often provide average flow rate conditions on portions of the domain's boundary. In engineering applications a common practice to complete these conditions is to prescribe a Dirichlet condition by assuming a priori a spatial profile for the velocity field. However, this strongly influences the accuracy of the numerical solution. A more mathematically sound approach is to prescribe the flow rate conditions using Lagrange multipliers, resulting in an augmented weak formulation of the Navier-Stokes problem.

In this paper we start from the SIMPLE preconditioner, introduced so far for the standard Navier-Stokes equations, and we derive two preconditioners for the monolithic solution of the augmented problem. This can be useful in complex applications where splitting the computation of the velocity/pressure and Lagrange multipliers numerical solutions can be very expensive. In particular, we investigate the numerical performance of the preconditioners in both idealized and real-life scenarios. Finally, we also highlight the advantages of treating flow rate conditions with a Lagrange multipliers approach instead of prescribing a Dirichlet condition.

AMS subject classifications: 65M22, 68U20, 76D05

Key words: Flow rate conditions, Lagrange multipliers, SIMPLE preconditioner, incompressible Navier-Stokes equations, computational fluid-dynamics.

1 Introduction

Incompressible Navier-Stokes equations are commonly used in engineering applications to study the dynamics of viscous fluids. Given a bounded domain $\Omega \subset \mathbb{R}^3$, classical boundary conditions impose three (one for each spatial dimension) point-wise data on

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the domain's boundary $\partial\Omega$, typically prescribing the components of the velocity field (Dirichlet condition), of the Cauchy normal stress (Neumann condition), or a combination of the two (Robin condition).

On the other hand, in some applications the only information available provides average conditions which are not enough to close the problem and thus need to be completed [11]. We will refer to them as *defective* boundary conditions. Some examples arise in blood-dynamics simulations, where clinical measures often provide information on either the *flow rate* or the *mean pressure* [46, 53], and flow in pipes simulations, where sensors measure the fluid flow rate [8, 43]. Moreover, in the *geometric multiscale approach* [44], where mathematical models with different spatial dimensions are coupled, the information provided by the lower-dimensional problem is not enough to close the higher-dimensional one [7, 38].

Regarding defective flow rate conditions, a widely employed strategy to close the problem, thanks to its practicality, is to prescribe a Dirichlet condition by assuming a-priori a spatial profile for the velocity field. In this case, the computational domain is often extended to reduce the impact of the chosen profile on the accuracy of the numerical solution. A more mathematically sound approach was proposed in [34], where the authors introduced a suitable variational formulation of the problem which includes the given flow rate data. However, this approach requires the definition of non-standard finite-dimensional subspaces which makes it problematic to implement in practice [54]. Hence, several alternative approaches have been proposed in the literature [28] based on Lagrange multipliers [25, 53], control theory [26, 37] and the Nitsche method [55, 58].

The Lagrange multiplier approach was proposed in [25] and applied to a quasi-Newtonian Stokes problem [21], in a fluid-structure interaction framework [27] and in practical hemodynamic problems with patient-specific geometries [12, 56]. In this approach, flow rate boundary conditions are considered as a constraint for the solution and enforced using Lagrange multipliers, resulting in an augmented weak formulation of the Navier-Stokes problem. The problem is closed by assuming that, on the considered portion of the domain's boundary, the Cauchy normal stress has zero tangential components and its normal component is constant in space. The resulting augmented problem can be solved by splitting the computation of the velocity/pressure fields and of the Lagrange multipliers in order to resort to available standard solvers for the solution of the Navier-Stokes step [53, 54]. However, iterative procedures based on this splitting can be very expensive in complex applications. In the recent work [35], the authors proposed monolithic block preconditioners based on inexact factorizations to deal with defective conditions arising from the coupling with lumped parameter models.

In order to efficiently solve the augmented Finite Element system, in this paper we consider a monolithic strategy and we propose a suitable block preconditioner for its efficient solution. Specifically, we start from the SIMPLE (Semi-Implicit Method for Pressure Linked Equations) iterative solution strategy [41], studied as a preconditioner in [18, 19, 39, 42], and we extend it to the augmented flow rate defective case. To test the effectiveness of our proposal, we present several numerical results in both idealized and