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Adaptive Neural Network Basis Methods for Partial Differential Equations with Low-Regular Solutions

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Abstract. This paper aims to devise an adaptive neural network basis method for numerically solving a second-order semi-linear partial differential equation with low-regular solutions in two/three dimensions. The method is obtained by combining basis functions from a class of shallow neural networks and the resulting multi-scale analogues, a residual strategy in adaptive methods and the non-overlapping domain decomposition method. Firstly, based on the solution residual, the domain Ω is iteratively decomposed and eventually partitioned into $K+1$ non-overlapping subdomains, denoted respectively as $\{\Omega_k\}_{k=0}^{K}$, where the exact solution is smooth on subdomain Ω_0 and low-regular on subdomain Ω_k ($1 \le k \le K$). Secondly, the low-regular solutions on different subdomains Ω_k ($1 \le k \le K$) are approximated by neural networks with different scales, while the smooth solution on subdomain Ω_0 is approximated by the initialized neural network. Thirdly, we determine the undetermined coefficients by solving the linear least squares problems directly or the nonlinear least squares problem via the Gauss-Newton method. The proposed method can be extended to multi-level case naturally. Finally, we use this adaptive method for several peak problems in two/three dimensions to show its high-efficient computational performance.

AMS subject classifications: 35R99, 65N22, 68T99

Key words: Neural network basis functions, domain decomposition, low-regular solutions, least squares problem, adaptive method.

1 Introduction

In the past few years, the machine learning including neural networks and artificial intelligence (AI) has experienced rapid and significant advances in computer science and

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data analysis (cf. [16, 27]). This technology has represented a cornerstone of innovation across various industries, continues to transform how we live and work from enhancing everyday experiences to driving groundbreaking discoveries. More recently, the machine learning method has also become an important approach to numerically solving partial differential equations (PDEs), which is a heart topic in computational and applied mathematics. Initial exploration of such study can be traced back to significant works in the 1990s (cf. [7, 26]). The typical methods along this line include but not limited to the following methods. The Deep Ritz method was introduced in [13] for solving elliptic problems by combining the Ritz method and deep neural networks (DNNs). The weak adversarial networks [44] is a machine learning method for solving a weak solution of a PDE based on its weak formulation and using the solution ansatz by DNNs (see also [4]). Another important classes of methods are the so-called Physics-Informed Neural Network (PINN) methods and their variants; we refer the reader to [6,23,25,30,32,33,36] for details. The applications of such methods in computational mathematics can be found in (cf. [12]). In all of the above machine learning methods, DNNs are used to parameterize the solution of PDEs and the related parameters are identified by minimizing an optimization problem formulated from the PDEs. The remarkable advantage of such methods connecting with DNNs is that they can overcome the so-called "curse of dimensionality" (cf. [1,2,17,22,24]). Furthermore, to improve computational efficiency, adaptive sampling techniques are combined with the PINN approach for solving PDEs with low-regular solutions (cf. [14, 15, 30, 38, 42]). Moreover, such techniques are also combined with the deep Ritz method for solving PDEs in [39].

Despite the DNN-based machine learning method can get over "curse of dimensionality", when solving PDEs in low dimensions, one is tempted to use shallow neural network as the solution asatz to balance computational accuracy and cost. Moreover, the weight/bias coefficients in the hidden layer are pre-set to random values and fixed, while the training parameters consist of the weight coefficients of output layer and they are obtained by using existing linear or nonlinear least squares solvers. In other words, one can use the shallow network to produce a basis of the underlying admissible space of the infinite-dimensional minimization problem associated with a PDE under discussion. An important advantage of such a basis is that it is mesh-free. We mention that randomness has been used in neural networks for a long time (cf. [35]). Randomized neural networks can be traced back to Turing's unorganized machine and Rosenblatt's perception [34, 41]. Later on, extreme Learning machine (ELM) was proposed in [18–20] for solving linear classification or regression problems by combining the previous random neural network with the linear least squares method. This method was also used for solving ordinary differential equations (cf. [28, 43]) and partial differential equations (PDEs) (cf. [11, 37]). The ELM method was further combined with the non-overlapping domain decomposition technique to solve PDEs effectively (cf. [8, 9]). In [40], the method was even applied to solve high-dimensional partial differential equations. The combination of the random neural network and the partition of unity technique gives rise to the random feature method (cf. [5]). More recently, transferable neural networks (cf. [45]) generate hidden