Commun. Comput. Phys.
doi: 10.4208/cicp.291109.080410s

Vol. **9**, No. 3, pp. 807-827
March 2011

# On the Order of Accuracy and Numerical Performance of Two Classes of Finite Volume WENO Schemes

Rui Zhang[1], Mengping Zhang[1] and Chi-Wang Shu[2,*]

[1] *Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, China.*
[2] *Division of Applied Mathematics, Brown University, Providence, RI 02912, USA.*

Received 29 November 2009; Accepted (in revised version) 8 April 2010

Available online 17 September 2010

*Dedicated to the memory of Professor David Gottlieb*

**Abstract.** In this paper we consider two commonly used classes of finite volume weighted essentially non-oscillatory (WENO) schemes in two dimensional Cartesian meshes. We compare them in terms of accuracy, performance for smooth and shocked solutions, and efficiency in CPU timing. For linear systems both schemes are high order accurate, however for nonlinear systems, analysis and numerical simulation results verify that one of them (Class A) is only second order accurate, while the other (Class B) is high order accurate. The WENO scheme in Class A is easier to implement and costs less than that in Class B. Numerical experiments indicate that the resolution for shocked problems is often comparable for schemes in both classes for the same building blocks and meshes, despite of the difference in their formal order of accuracy. The results in this paper may give some guidance in the application of high order finite volume schemes for simulating shocked flows.

**AMS subject classifications**: 65M08

**Key words**: Weighted essentially non-oscillatory (WENO) schemes, finite volume schemes, accuracy.

## 1 Introduction and the setup of the schemes

In this paper we are interested in numerically solving two dimensional conservation law systems

$$u_t + f(u)_x + g(u)_y = 0 \tag{1.1}$$

*Corresponding author. *Email addresses:* rui@ustc.edu.cn (R. Zhang), mpzhang@ustc.edu.cn (M. Zhang), shu@dam.brown.edu (C.-W. Shu)

 

with suitable initial and boundary conditions, using the finite volume schemes on Cartesian meshes. For this purpose, the computational domain is decomposed to rectangular cells

$$\Omega_{ij}=[x_{i-1/2},x_{i+1/2}]\times[y_{j-1/2},y_{j+1/2}],$$

and for simplicity we assume the mesh sizes $\Delta x=x_{i+1/2}-x_{i-1/2}$ and $\Delta y=y_{j+1/2}-y_{j-1/2}$ are constants. This assumption is not essential: finite volume schemes in this paper can be defined on arbitrary Cartesian meshes, even those with abrupt changes in mesh sizes, without affecting their conservation, accuracy and stability, in contrast to high order conservative finite difference schemes which can only be defined on smooth meshes. Finite volume schemes are also easier to implement in an adaptive mesh environment, for example in the AMR type schemes (e.g., [18]). This is the main reason that high order finite volume schemes are still commonly used in practice, even though high order finite difference schemes are much less expensive in multi-dimensions in uniform or smooth Cartesian meshes, see for example [3] for a comparison of finite volume and finite difference schemes in the context of essentially non-oscillatory (ENO) reconstructions.

In a finite volume scheme we seek approximations to the cell averages

$$\bar{\tilde{u}}_{i,j}=\frac{1}{\Delta x\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}u(x,y)dxdy. \tag{1.2}$$

We use the notation $\bar{u}$ to denote the cell averaging operation in the $x$-direction (integral in the cell $[x_{i-1/2},x_{i+1/2}]$ divided by the cell size $\Delta x$), and $\tilde{u}$ to denote the cell averaging operation in the $y$-direction. The two dimensional cell average $\bar{\tilde{u}}$ can be obtained by successively performing the cell averaging operators in $x$ and in $y$. If we integrate the conservation law (1.1) over the cell $\Omega_{ij}$ and then divide by its area, we obtain

$$\frac{d\bar{\tilde{u}}_{i,j}}{dt}+\frac{1}{\Delta x}\left(\tilde{f}_{i+\frac{1}{2},j}-\tilde{f}_{i-\frac{1}{2},j}\right)+\frac{1}{\Delta y}\left(\bar{g}_{i,j+\frac{1}{2}}-\bar{g}_{i,j-\frac{1}{2}}\right)=0, \tag{1.3}$$

where $\bar{\tilde{u}}_{i,j}$ is the cell average (1.2) and

$$\tilde{f}_{i+\frac{1}{2},j}=\frac{1}{\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}f\left(u(x_{i+\frac{1}{2}},y)\right)dy, \tag{1.4a}$$

$$\bar{g}_{i,j+\frac{1}{2}}=\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}g\left(u(x,y_{j+\frac{1}{2}})\right)dx \tag{1.4b}$$

are the physical fluxes, which are cell averages of $f(u)$ in $y$ at $x=x_{i+1/2}$ and of $g(u)$ in $x$ at $y=y_{j+1/2}$ respectively. Although (1.3) looks like a scheme, we should emphasize that it is actually an equality satisfied by the exact solution of the PDE (1.1).

Notice that the equality (1.3) describes the evolution of the cell averages $\bar{\tilde{u}}_{i,j}$ while requiring the information of point values of the solution $u$ in evaluating the physical fluxes in (1.4a) and (1.4b). In order to convert the equality (1.3) to a scheme (commonly