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A Causality-DeepONet for Causal Responses of Linear Dynamical Systems

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Abstract. In this paper, we propose a DeepONet structure with causality to represent causal linear operators between Banach spaces of time-dependent signals. The theorem of universal approximations to nonlinear operators proposed in [5] is extended to operators with causalities, and the proposed Causality-DeepONet implements the physical causality in its framework. The proposed Causality-DeepONet considers causality (the state of the system at the current time is not affected by that of the future, but only by its current state and past history) and uses a convolution-type weight in its design. To demonstrate its effectiveness in handling the causal response of a physical system, the Causality-DeepONet is applied to learn the operator representing the response of a building due to earthquake ground accelerations. Extensive numerical tests and comparisons with some existing variants of DeepONet are carried out, and the Causality-DeepONet clearly shows its unique capability to learn the retarded dynamic responses of the seismic response operator with good accuracy.

AMS subject classifications: 65R20, 65Z05, 78M25

Key words: Neural network, universal approximation theory of nonlinear operator, DeepONet, Causality-DeepONet.

1 Introduction

Computing operators between physical quantities defined in function spaces have many applications in forward and inverse problems in scientific and engineering computations. For example, in wave scattering in inhomogeneous or random media, the mapping between the media physical properties, which can be modelled as a random field, and the

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wave field is a nonlinear operator, which represents some of the most challenging computational tasks in medical imaging, geophysical and seismic problems. A specific example comes from earthquake safety studies of buildings and structures, the response of structures to seismic ground accelerations gives rise to a causal operator between spaces of highly oscillatory temporal signals.

Structural dynamic analysis has always been one of the crucial problems in the civil engineering field. Traditionally, researchers in this field analyzing structural dynamic response focus on constructing proper mathematical models like ordinary or partial differential equations and utilizing grid-based numerical methods to solve them. The finite element method [50] is one of the popular methods considered for the solutions along with an appropriate time integration scheme like Newmark's-Beta method [6, 36]. Alternatively, system identification-based methods, as an attempt to construct a surrogate model by mapping the input signals to the output responses directly, have shown their superior capability in accelerating the computations. A comprehensive review of this approach was provided in [42, 19]. Meanwhile, recently learning time sequential response operator between input and output signals [23] has been studied using recurrent neural network (RNN) [11,24], long short-term memory neural network (LSTM) [17], WaveNet [37], the one-step ResNet approximation [40] and the multi-step recurrent ResNet approximation [40]. The RNN and its variant LSTM are ubiquitous network structures for predicting time series in financial engineering, machine translation, and sentiment analysis and so on in the natural language processing field. In particular, LSTM has been shown to have the potential to predict building responses excited by seismic ground accelerations [20, 46]. The one-step ResNet approximation and the multi-step recurrent ResNet approximation, provide the approximation to the integral form of the dynamical system and have been demonstrated effective equation recovery for linear and nonlinear dynamical systems [12, 40].

Deep neural networks (DNNs), as one of the most intuitive frameworks for model reductions with its superior ability to approximate general high dimensional functions [7], have been considered recently in learning mappings whose closed forms are not known. So far, DNNs have shown much promise in solving problems from scientific and engineering computing, including initial and boundary value problems of ODEs and PDEs [4, 10, 15, 18, 30, 41, 47, 28]. Soon after universal approximation theorems to functions by neural networks was proposed [7], Chen & Chen [5] proved that there also exists a framework that could give universal approximations to nonlinear operators between Banach spaces. Based on this theory, the DeepONet [32] was constructed for learning operators where trunk net functions are used as basis and the branch net functions as mappings from the input functions to some hidden manifolds. The DeepONet replaced the one-hidden layer networks in the original proposal in Chen & Chen's paper [5] by two deep neural networks, which has been shown to have the potential to break the curse of dimensionality from the input space. In the meantime, other approach for learning operators based on a graph kernel network [26] for PDEs has also been proposed. The nonlin-