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Finite Difference Approximation with ADI Scheme for Two-Dimensional Keller-Segel Equations

Yubin Lu, Chi-An Chen, Xiaofan Li* and Chun Liu

Department of Applied Mathematics, College of Computing, Illinois Institute of Technology, Chicago, IL 60616, USA.

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Abstract. Keller-Segel systems are a set of nonlinear partial differential equations used to model chemotaxis in biology. In this paper, we propose two alternating direction implicit (ADI) schemes to solve the 2D Keller-Segel systems directly with minimal computational cost, while preserving positivity, energy dissipation law and mass conservation. One scheme unconditionally preserves positivity, while the other does so conditionally. Both schemes achieve second-order accuracy in space, with the former being first-order accuracy in time and the latter second-order accuracy in time. Besides, the former scheme preserves the energy dissipation law asymptotically. We validate these results through numerical experiments, and also compare the efficiency of our schemes with the standard five-point scheme, demonstrating that our approaches effectively reduce computational costs.

AMS subject classifications: 65M06, 35K61,35K55, 65Z05

Key words: Keller-Segel equations, energy dissipation, positive preserving, ADI scheme.

1 Introduction

The Keller-Segel system as a mathematical model was proposed to model the phenomenon of chemotaxis in biology, established by Patlak [26] and Keller and Segel [17,18]. Chemotaxis is the migration of organisms in response to chemical signals. Usually, the organisms are attracted by chemical signals that could be emitted by themselves. On the other hand, there is another hidden mechanism, namely the diffusion phenomenon. These two mechanisms compete with each other, leading to different interesting biological phenomena. These phenomena could be summarized by the Keller-Segel equations, a system of parabolic-parabolic equations or parabolic-elliptic equations in a limiting case. So far,

*Corresponding author. *Email addresses:* lix@iit.edu (X. Li), ylu117@iit.edu (Y. Lu), cchen156@hawk.iit.edu (C.-A. Chen), cliu124@iit.edu (C. Liu)

the Keller-Segel equations are not limited to describing such biological processes, but can also be widely used to describe various phenomena in social sciences and other competitive systems.

One of the most important mathematical issues of the Keller-Segel system is the well-posedness of their solutions. This issue has been well studied by mathematicians in some sense [2, 4, 14, 15]. Roughly speaking, if the initial mass is less than a critical value, the solution exists globally, otherwise the solution will blow up in finite time. Certainly, blow-up does not occur in real scenarios. This indicates that the classical Keller-Segel model has limitations. Therefore, several improved models are designed to relax such restrictions [27]. We will focus on the classical one.

In addition to the analytic aspect of the Keller-Segel equations, the numerical aspect is also extremely important for real applications. The Keller-Segel system consists of two equations, one of which describes the evolution of density of organisms and the other describes the evolution of the chemoattractant concentration. Naturally, the density and the concentration should remain nonnegative, the density should also preserve mass conservation under suitable boundary conditions, and the energy dissipation law holds. A question arises: how do we design numerical solvers that preserve these properties? In addressing this question, many works have been proposed to solve such type equations with specific conditions. The most popular methods including linear/nonlinear finite volume schemes [1, 3, 5, 6, 19, 34], discontinuous Galerkin schemes [10, 20], discontinuous finite element scheme [9], finite difference scheme [25], spectral method [28], and others [8, 16]. Moreover, several general ways have been proposed to preserve the energy dissipation law for specific problems, including scalar auxiliary variable approach (SAV) [30], convex splitting method [7], stabilization method [31, 35], and the method of invariant energy quadratization (IEQ) [32, 33]. Furthermore, numerous numerical schemes have been developed to address the challenges posed by the Poisson-Nernst-Planck equations. Given the interrelated nature of Keller-Segel equations and Poisson-Nernst-Planck equations in the context of numerical scheme design, we recommend that readers consult [21, 23, 24, 29], and the references cited therein for more insights. Although these numerical methods are developed to preserve positivity, mass conservation and energy dissipation and so on, very few of the existing papers take into account the reduction of computational cost while still preserving these desired properties.

In this work, we consider the following 2D Keller-Segel equations

$$\partial_t \rho = \Delta \rho - \nabla \cdot (\rho \nabla c), \tag{1.1}$$

$$\varepsilon \partial_t c = \Delta c + \rho, \tag{1.2}$$

$$\rho(\mathbf{x},0) = f(\mathbf{x}), \quad c(\mathbf{x},0) = g(\mathbf{x}),$$

where ρ and c are the cell density and the chemoattractant concentration respectively, $\mathbf{x} = (x,y) \in \Omega \subset \mathbb{R}^2$, and ε is a nonnegative constant. To preserve positivity, Liu *et al*. [25] reformulated the density equation (1.1) as the following symmetric form:

$$\partial_t \rho = \nabla \cdot \left(e^c \nabla \left(\frac{\rho}{e^c} \right) \right), \tag{1.3}$$