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Large Time Behavior of Solutions to Burgers Equation with a Time Delay

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Abstract. In this article, we consider the large time behavior of solutions ρ to the one-dimensional Cauchy problem of the generalized viscous Burgers equations with a time delay. More precisely, we derive the time decay estimate of the solution to the Cauchy problem, provided by $\rho_0(0)\in L^1$ and $\rho_0\in C([-\tau,0];H^1)$. The result is based on the combination of the weighted L^2 energy estimate and the L^1 estimate.

AMS subject classifications: 35B40, 35Q35

Key words: Burgers equation, time delay, large time behavior.

1 Introduction

In this article, we shall focus on the one-dimensional generalized Burgers equation with a time delay

$$\partial_t\rho-\nu\partial_x^2\rho+\partial_x\big(\rho V(\rho_\tau)\big)=0 \tag{1.1}$$

for $t>0$ and $x\in\mathbb{R}$. Here, ρ is an unknown function, ν is a positive constant, $V(\rho)$ denotes a given function which depends on ρ, and $\rho_\tau(t):=\rho(t-\tau)$.

For (1.1), we consider the initial history problem with the initial history

$$\rho(\theta,x)=\rho_0(\theta,x) \tag{1.2}$$

for $-\tau\le\theta\le 0$ and $x\in\mathbb{R}$, where $\rho_0=\rho_0(\theta,x)$ is a the given function.

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As is well-known, a simple model of traffic flow is given by the following viscous Burgers equation:

$$\partial_t\rho-\nu\partial_x^2\rho+\partial_x\left\{V_m\rho\left(1-\frac{\rho}{\rho_m}\right)\right\}=0, \tag{1.3}$$

where V_m and ρ_m are positive constants denote the maximum speed as $\rho\to 0$ and the maximum density, respectively. It is also well-known that the viscous Burgers equations (1.3) is a strictly parabolic partial differential equation, so its solution has an infinite propagation speed. In order to improve this troubling feature, the authors suggested the viscous Burgers equation (1.1) with a time delay in [2]. In [2], the authors proved the existence of the global in time solution under some conditions for the delay parameter and the initial history. Moreover, they showed the result concerning the regularity property of global in time solutions.

Here, we introduce the known results concerning our problem. At first, we shall focus on the viscous Burgers equations whose nonlinear term has a time delay. Liu [3] considered the asymptotic behavior of the solutions to (1.1) with $V(\rho_\tau)=\rho_\tau$ under zero-Dirichlet boundary condition. He proved the unique existence theorem of the global mild solution. Moreover, Tang and Wang [7] considered the same problem as in [3] and proved the exponential stability of the solution if the time delay parameter is sufficiently small. In the periodic boundary condition case, Smaoui and Mekkaoui [5] proved similar results. Tang [6] also considered the following viscous Burgers equation with a nonlocal delay effect:

$$\partial_t\rho-\nu\partial_x^2\rho+(\varphi,\rho_\tau)\partial_x\rho=0$$

with zero-Dirichlet boundary condition, where (φ,ρ_τ) denotes L^2 product of given L^2 function φ and delay term ρ_τ. He proved that the solution is exponentially stable, provided that the delay term is sufficiently small.

On the other hand, there are many results for the viscous Burgers equation without a time delay. It is well-known that the viscous Burgers equation has the global in time solution for the initial data $\rho_0\in H^1$. Furthermore, if the initial data satisfy $\rho_0\in H^1\cap L^1$, the global in time solution of the viscous Burgers equation has the following asymptotic behavior:

$$\left\|\partial_x^k\rho(t)\right\|_{L^2}=\mathcal{O}(t^{-1/4-k/2}) \tag{1.4}$$

for $k=0,1$ (see [4], for example).

In this article, our main purposes are to improve the results of the existence of the global in time solution obtained in [2], and derive the decay estimate of the solution to the Cauchy problem (1.1), (1.2), which is similar to (1.4), under the assumptions $\rho_0(0)\in L^1$ and $\rho_0\in C([-\tau,0];H^1)$.